The entropy function on an algebraic structure with infinite partition and $m-$preserving transformation generators

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Abstract. One of the applied branches of mathematics is the entropy of a dynamical system. In this paper we defined the infinite partition of an algebraic structure and then we introduce the entropy of a countable partition of this structure. In this respect, we introduce the generators of an m-preserving transformation of a discrete dynamical system. At the end, we prove a version of Kolomogorov-Sinai theorem concerning the entropy of a dynamical system.

M.S.C. 2000: 28D20, 37A35, 54C70, 37-XX.
Key words: F-measure, countable partition, entropy, m-preserving transformation, generators.

1 Introduction

We assume that the reader is familiar with the definition of dynamical systems and ergodic theory. In physics, entropy of a system with a finite quantum states is defined by $S = -k \sum_v f_v \ln(f_v); \sum_v f_v = 1$, where k is the Boltzmann constant and the sum is over all quantum states. This formula can be interpreted as a degree of disordering of the system. The entropy of an algebraic structure with a finite partition was defined by Riecăn [6]. In this paper we will extend this notion to an algebraic structure with infinite partition and we discuss ergodic theory properties.

2 Basic concepts

Let F be a non-empty totally ordered set. Also let $\oplus$, $\odot$ be two binary operations on F and 1 be a constant element of F such that,

$$1 \odot a = a \geq a \odot b. \tag{2.1}$$

Definition 2.1. A function $m : F \longrightarrow [0, 1]$ is called F-measure if for any a, b and $c \in F$ we have,

Applied Sciences, Vol.12, 2010, pp. 48-63.

(i) $m(a \oplus b) = m(b \oplus a)$, $m(a \odot b) = m(b \odot a)$;

(ii) $m(a \oplus (b \oplus c)) = m((a \oplus b) \oplus c)$, $\quad m(a \odot (b \odot c)) = m((a \odot b) \odot c)$;

(iii) $m(a \odot (b \oplus c)) = m((a \odot b) \oplus (a \odot c))$, $\quad m(a \oplus (b \odot c)) = m((a \oplus b) \odot (a \oplus c))$;

(iv) $m(\oplus_{i=1}^{n} a_i) = \sum_{i=1}^{n} m(a_i)$, for any $n \in \mathbb{N}$;

(v) If $a \leq b$ then $m(a) \leq m(b)$;

(vi) $m(a \odot b) \leq m(a)$;

(vii) If $m(a) = m(1)$ then $m(a \odot b) = m(b)$;

(viii) If $m(a) \leq m(b)$ then $m(a \odot c) \leq m(b \odot c)$.

Definition 2.2. A countable partition in F is a sequence $\mathscr{A} = \{a_i\}_{i \in \mathbb{N}} \subseteq F$ such that,

(i) $m(1) = \sum_{i=1}^{\infty} m(a_i)$;

(ii) $\sum_{i=1}^{\infty} m(a_i \odot b) = m(b)$, for any $b \in F$.

Remark 2.3. It is clear that if a countable partition $\mathscr{A} = \{a_i\}_{i \in \mathbb{N}}$ has more than one element, then

$$m(a_i) < m(1),$$

for any $i \in \mathbb{N}$. And therefore,

$$\sum_{i=1, i \neq n}^{\infty} m(a_i) < m(1),$$

for any $n \in \mathbb{N}$.

Definition 2.4. Let $\mathscr{A} = \{a_i\}_{i \in \mathbb{N}}$ and $\mathscr{B} = \{b_j\}_{j \in \mathbb{N}}$ be two countable partitions in F. Their join is

$$\mathscr{A} \nabla \mathscr{B} = \{a_i \odot b_j : a_i \in \mathscr{A}, b_j \in \mathscr{B}, i, j \in \mathbb{N}\},$$

if $\mathscr{A} \neq \mathscr{B}$, and

$$\mathscr{A} \nabla \mathscr{A} = \mathscr{A}.$$

Definition 2.5. Let $\mathscr{A} = \{a_i\}_{i \in \mathbb{N}}$ and $\mathscr{B} = \{b_j\}_{j \in \mathbb{N}}$ be two countable partitions in F. We say $\mathscr{B}$ is a refinement of $\mathscr{A}$, and write $\mathscr{A} \prec \mathscr{B}$, if

(i) For each $a_i \in \mathscr{A}$ there exists $b_{i_1}, \ldots, b_{i_{n_i}} \in \mathscr{B}$ such that,

$$m(a_i) = \sum_{j=1}^{n_i} m(b_{i_j});$$

(ii) If a_i, a_k are two distinct element of $\mathscr{A}$ such that $m(a_i) = \sum_{j=1}^{n_i} m(b_{i_j})$ and $m(a_k) = \sum_{j=1}^{n_k} m(b_{k_j})$, then $b_{i_j} \neq b_{k_l}$ for any $j \in \{1, \ldots, n_i\}$ and $l \in \{1, \ldots, n_k\}$.

Proposition 2.6. *$\mathscr{A} \nabla \mathscr{B}$ with Lexicographic ordering is a countable partition in F.*

Proof. Since $\mathscr{A}\nabla\mathscr{B}$ is given Lexicographic order, we have,

$$\sum_{i,j\in\mathbb{N}} m(a_i \odot b_j) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} m(a_i \odot b_j).$$

Therefore,

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} m(a_i \odot b_j) = \sum_{j=1}^{\infty} m(b_j) = m(1).$$

So these imply that,

$$\sum_{i,j\in\mathbb{N}} m(a_i \odot b_j) = m(1).$$

On the other hand, for any $c \in F$ we have,

$$\begin{aligned} \textstyle\sum_{i,j\in\mathbb{N}} m((a_i \odot b_j) \odot c) &= \textstyle\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} m(a_i \odot (b_j \odot c)) \\ &= \textstyle\sum_{j=1}^{\infty} m(b_j \odot c) = m(c). \end{aligned}$$

□

3 Entropy of a countable partition in F

Definition 3.1. Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ be a countable partition in F. The entropy of $\mathscr{A}$ is defined by

$$H(\mathscr{A}) = -\log \sup_{i\in\mathbb{N}} m(a_i).$$

Proposition 3.2. *Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ and $\mathscr{B} = \{b_j\}_{j\in\mathbb{N}}$ be two countable partitions in F. If $\mathscr{B}$ is a refinement of $\mathscr{A}$, then for any $b_k \in \mathscr{B}$ there exist some $a_l \in \mathscr{A}$ such that*

$$m(b_k) \leq m(a_l).$$

Proof. If it isn't so, we assume that there exists $k_0 \in \mathbb{N}$ such that

$$m(b_{k_0}) > m(a_i)$$

for any $i \in \mathbb{N}$.
Since $\mathscr{A} \prec \mathscr{B}$, for any $a_i \in \mathscr{A}$, there exist $b_{i_j} \in \mathscr{B}$ and $n_i \in \mathbb{N}$ such that

$$m(a_i) = \sum_{i=1}^{n_i} m(b_{i_j}).$$

So these imply that

$$m(b_{k_0}) > \sum_{i=1}^{n_i} m(b_{i_j}).$$

Since $\mathscr{B}$ is a refinement of $\mathscr{A}$ we have

$$\sum_{i=1}^{\infty} m(a_i) = \sum_{i=1}^{\infty}\sum_{j=1}^{n_i} m(b_{i_j}). \tag{3.1}$$

But, we know that for any $i \in \mathbb{N}, j \in \{1, \dots, n_i\}$, we have $b_{k_0} \neq b_{i_j}$.
Since otherwise, we would have two cases:

case(1) If $n_i = 1$, then $m(b_{k_0}) > m(b_{k_0})$, and this is a contradiction.

case(2) If $n_i \geq 2$, then $0 > \sum_{j=1, i_j \neq k_0}^{n_i} m(b_{i_j})$, and this is also a contradiction.

Now, since $\mathscr{A}$ is a countable partition in F, in the right side of the Equation (3.1) we have all $b_{i_j} \in \mathscr{B}$ except for b_{k_0}, and there are no repeated b_{i_j}. Then

$$m(1) = \sum_{j=1, j \neq k_0}^{\infty} m(b_j).$$

But since $\mathscr{B}$ is a countable partition in F, this contradicts the Remark 2.3. □

Proposition 3.3. *Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ and $\mathscr{B} = \{b_j\}_{j\in\mathbb{N}}$ be two countable partitions in F.If $\mathscr{B}$ is a refinement of $\mathscr{A}$, then $H(\mathscr{A}) \leq H(\mathscr{B})$.*

Proof. It is clear by using Proposition 3.2. □

Definition 3.4. Two countable partitions $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ and $\mathscr{B} = \{b_j\}_{j\in\mathbb{N}}$ in F are called independent if

$$m(a_i \odot b_j) = m(a_i)m(b_j)$$

for any $i, j \in \mathbb{N}$.

Proposition 3.5. *Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ and $\mathscr{B} = \{b_j\}_{j\in\mathbb{N}}$ be two countable partitions in F.Then*

(i) $H(\mathscr{A}\nabla c) \geq H(\mathscr{A})$, for any $c \in F$

(ii) $H(\mathscr{A}\nabla\mathscr{B}) \geq H(\mathscr{A})$ and $H(\mathscr{A}\nabla\mathscr{B}) \geq H(\mathscr{B})$;

(iii) If $\mathscr{A}$ and $\mathscr{B}$ are independent, then

$$H(\mathscr{A}\nabla\mathscr{B}) = H(\mathscr{A}) + H(\mathscr{B}).$$

Proof.

(i) For any $i \in \mathbb{N}$ we have $a_i \geq a_i \odot c$. Therefore

$$m(a_i) \geq m(a_i \odot c),$$

for any $i \in \mathbb{N}$.

(ii) We have $m(a_i) \geq m(a_i \odot b_j)$, for any $i, j \in \mathbb{N}$.
This follows that

$$\sup_{i\in\mathbb{N}} m(a_i) \geq \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} m(a_i \odot b_j).$$

Therefore,

$$H(\mathscr{A}\nabla\mathscr{B}) \geq H(\mathscr{A}).$$

Similarly we have, $H(\mathscr{A}\nabla\mathscr{B}) \geq H(\mathscr{B})$.

(iii)

$$\begin{aligned} H(\mathscr{A}\nabla\mathscr{B}) &= -\log \sup_{i,j\in\mathbb{N}} m(a_i \odot b_j) \\ &= -\log \sup_{i,j\in\mathbb{N}} m(a_i)m(b_j) \\ &= H(\mathscr{A}) + H(\mathscr{B}). \end{aligned}$$

□

Definition 3.6. Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ be a countable partition in F and $c \in F$ such that $m(c) \neq 0$. The conditional entropy of $\mathscr{A}$ given c is defined by

$$H(\mathscr{A} \mid c) = -\log \sup_{i\in\mathbb{N}} m(a_i \mid c),$$

where $m(a_i \mid c) = \dfrac{m(a_i \odot c)}{m(c)}$.

Proposition 3.7. *Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ and $\mathscr{B} = \{b_j\}_{j\in\mathbb{N}}$ be two countable partitions in F.Let c and d be two arbitrary elements in F, such that $m(c) \neq 0$ and $m(d) \neq 0$. We have*

(i) $H(\mathscr{A}\nabla c) \geq H(\mathscr{A} \mid c)$;

(ii) *If* $d \leq c$, *then* $H(\mathscr{A}\nabla d) \leq H(\mathscr{A}\nabla c)$;

(iii) *If* $\mathscr{A} \prec \mathscr{B}$, *then* $H(\mathscr{A} \mid c) \leq H(\mathscr{B} \mid c)$.

Proof.

(i) Since $m(c) \in (0, 1]$, we have

$$m(a_i \odot c) \leq \frac{m(a_i \odot c)}{m(c)}.$$

It follows that

$$-\log \sup_{i\in\mathbb{N}} m(a_i \odot c) \geq -\log \sup_{i\in\mathbb{N}} \frac{m(a_i \odot c)}{m(c)}.$$

(ii) Since $d \leq c$, we have $m(a_i \odot d) \leq m(a_i \odot c)$, for any $i \in \mathbb{N}$.

(iii) Since $\mathscr{A} \prec \mathscr{B}$, for any $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that

$$m(b_k) \leq m(a_l).$$

Then we have

$$m(b_k \odot c) \leq m(a_l \odot c).$$

□

Definition 3.8. The entropy function of $c \in F$ is defined by

$$H(c) = \begin{cases} -\log m(c) & \text{if} \quad m(c) > 0 \\ 0 & \text{if} \quad m(c) = 0 \end{cases}$$

Proposition 3.9. *Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ be a countable partition in F and c, d be arbitrary elements in F such that $m(c) \neq 0$. Then*

(i) $H(c) \geq 0$;

(ii) $H(\mathscr{A}\nabla c) \geq H(c)$;

(iii) If $d \leq c$, then $H(c) \leq H(d)$;

(iv) $H(\mathscr{A}\nabla c) = H(\mathscr{A} \mid c) + H(c)$.

Proof. $(i), (ii), (iii)$ are clear.

(iv)

$$\begin{aligned} H(\mathscr{A} \mid c) + H(c) &= -\log \sup_{i\in\mathbb{N}} \frac{m(a_i \odot c)}{m(c)} + (-\log m(c)) \\ &= -\log \sup_{i\in\mathbb{N}} \frac{m(a_i \odot c)}{m(c)}\ m(c). \end{aligned}$$

□

Definition 3.10. Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ be a countable partition in F. The diameter of $\mathscr{A}$ is defined as follows,

$$diam(\mathscr{A}) = \sup_{i\in\mathbb{N}} m(a_i).$$

Definition 3.11. Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ and $\mathscr{B} = \{b_j\}_{j\in\mathbb{N}}$ be two countable partitions in F. The conditional entropy of $\mathscr{A}$ given $\mathscr{B}$ is defined by,

$$H(\mathscr{A} \mid \mathscr{B}) = -\log \sup_{i\in\mathbb{N}} \frac{diam(a_i\nabla\mathscr{B})}{diam\mathscr{B}}.$$

Remark 3.12. (i) It is easy to see,

$$-\log \sup_{i\in\mathbb{N}} \frac{diam(a_i\nabla\mathscr{B})}{diam\mathscr{B}} = -\log \sup_{j\in\mathbb{N}} \frac{diam(\mathscr{A}\nabla b_j)}{diam\mathscr{B}}.$$

(ii) If we set $P^0 = \{1\}$, then P^0 is a countable partition in F and

$$H(\mathscr{A}|P^0) = H(\mathscr{A}).$$

Proposition 3.13. *Let $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$, $\mathscr{B} = \{b_j\}_{j\in\mathbb{N}}$ and $\mathscr{C} = \{c_k\}_{k\in\mathbb{N}}$ be countable partitions in F. We have,*

(i) $H(\mathscr{A} \mid \mathscr{B}) \geq 0$;

(ii) $H(\mathscr{A}\nabla\mathscr{B} \mid \mathscr{C}) = H(\mathscr{A} \mid \mathscr{C}) + H(\mathscr{B} \mid \mathscr{A}\nabla\mathscr{C})$;

(iii) $H(\mathscr{A}\nabla\mathscr{B}) = H(\mathscr{A}) + H(\mathscr{B} \mid \mathscr{A})$;

(iv) *If* $\mathscr{A} \prec \mathscr{B}$, *then* $H(\mathscr{A} \mid \mathscr{C}) \leq H(\mathscr{B} \mid \mathscr{C})$;

(v) *If* $\mathscr{B} \prec \mathscr{C}$, *then* $H(\mathscr{A} \mid \mathscr{B}) \leq H(\mathscr{A}\nabla\mathscr{C})$.

In particular, $H(\mathscr{A} \mid \mathscr{B}) \leq H(\mathscr{A}\nabla\mathscr{B})$;

(vi) $H(\mathscr{A}) \geq H(\mathscr{A} \mid \mathscr{B})$;

(vii) $H(\mathscr{A}\nabla\mathscr{B}) \leq H(\mathscr{A}) + H(\mathscr{B})$;

(viii) *If* $\mathscr{A}$ *and* $\mathscr{B}\nabla\mathscr{C}$ *are independent, then*

$$H(\mathscr{A}\nabla\mathscr{B} \mid \mathscr{C}) = H(\mathscr{A}) + H(\mathscr{B} \mid \mathscr{C}).$$

Proof.

(i) It is clear.

(ii)

$$\begin{aligned}
\sup_{k\in\mathbb{N}} \frac{diam((\mathscr{A}\nabla\mathscr{B})\nabla c_k)}{diam\ \mathscr{C}} &= \sup_{k\in\mathbb{N}} \frac{diam(\mathscr{A}\nabla c_k)}{diam\ \mathscr{C}} \times \frac{diam((\mathscr{A}\nabla\mathscr{B})\nabla c_k)}{diam(\mathscr{A}\nabla c_k)} \\
&= \sup_{k\in\mathbb{N}} \frac{\sup_{i\in\mathbb{N}} m(a_i \odot c_k)}{diam\ \mathscr{C}} \times \frac{\sup_{i,j\in\mathbb{N}} m(a_i \odot b_j \odot c_k)}{\sup_{i\in\mathbb{N}} m(a_i \odot c_k)} \\
&= (\sup_{k\in\mathbb{N}} \frac{\sup_{i\in\mathbb{N}} m(a_i \odot c_k)}{diam\ \mathscr{C}})(\frac{\sup_{i,j,k\in\mathbb{N}} m(a_i \odot b_j \odot c_k)}{\sup_{i,k\in\mathbb{N}} m(a_i \odot c_k)}).
\end{aligned}$$

Then,

$$\begin{aligned}
H(\mathscr{A}\nabla\mathscr{B} \mid \mathscr{C}) &= -\log \sup_{k\in\mathbb{N}} \frac{diam((\mathscr{A}\nabla\mathscr{B})\nabla c_k)}{diam\ \mathscr{C}} \\
&= -\log \sup_{k\in\mathbb{N}} \frac{diam(\mathscr{A}\nabla c_k)}{diam\ \mathscr{C}} - \log \sup_{j\in\mathbb{N}} \frac{diam(b_j\nabla(\mathscr{A}\nabla\mathscr{C}))}{diam(\mathscr{A}\nabla\mathscr{C})}
\end{aligned}$$

(iii) Set $\mathscr{C} = P^0$ in (i).

(iv) Since $\mathscr{A} \prec \mathscr{B}$, for any $j \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that,

$$m(b_j \odot c_k) \leq m(a_l \odot c_k).$$

Then,

$$\sup_{j,k\in\mathbb{N}} m(b_j \odot c_k) \leq \sup_{i,j\in\mathbb{N}} m(a_l \odot c_k).$$

(v) As $\mathscr{B} \prec \mathscr{C}$, for any $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that,

$$m(a_i \odot c_k) \leq m(a_i \odot b_l),$$

for any $i \in \mathbb{N}$.
Therefore,

$$\sup_{k\in\mathbb{N}} \sup_{i\in\mathbb{N}} m(a_i \odot c_k) \leq \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} m(a_i \odot b_j). \tag{3.2}$$

On the other hand, because $\mathscr{B}$ is a countable partition in F, there exists $j \in \mathbb{N}$ such that $m(b_j) > 0$. Therefore, $0 < diam\mathscr{B} \leq 1$. And it follows that,

$$\begin{aligned} \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} m(a_i \odot b_j) &\leq \frac{\sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} m(a_i \odot b_j)}{diam\mathscr{B}} \\ &= \sup_{j\in\mathbb{N}} \frac{\sup_{i\in\mathbb{N}} m(a_i \odot b_j)}{diam\mathscr{B}} \\ &= \sup_{j\in\mathbb{N}} \frac{diam(\mathscr{A}\nabla b_j)}{diam\mathscr{B}} \qquad (3.3) \end{aligned}$$

Then we have,

$$\sup_{i,k\in\mathbb{N}} m(a_i \odot c_k) \leq \sup_{j\in\mathbb{N}} \frac{diam(\mathscr{A}\nabla b_j)}{diam\mathscr{B}}.$$

vi) We have,

$$m(a_i) \geq m(a_i \odot b_j),$$

for any $i, j \in \mathbb{N}$. Therefore,

$$\sup_{i\in\mathbb{N}} m(a_i) \leq \sup_{i,j\in\mathbb{N}} m(a_i \odot b_j) \leq \frac{\sup_{i,j\in\mathbb{N}} m(a_i \odot b_j)}{diam\ \mathscr{B}}.$$

(vii) It follows immediately from (iii) and (vi).

(viii) $\mathscr{A}$ and $\mathscr{B}\nabla\mathscr{C}$ are independent, so

$$m(a_i \odot (b_j \odot c_k)) = m(a_i)m(b_j \odot c_k),$$

for any $i, j, k \in \mathbb{N}$.
Then,

$$\begin{aligned} H(\mathscr{A}\nabla\mathscr{B} \mid \mathscr{C}) &= -\log \sup_{k\in\mathbb{N}} \frac{\sup_{i,j\in\mathbb{N}} m(a_i \odot (b_j \odot c_k))}{diam\ \mathscr{C}} \\ &= -\log \sup_{k\in\mathbb{N}} \frac{\sup_{i,j\in\mathbb{N}} m(a_i)m(b_j \odot c_k)}{diam\ \mathscr{C}} \\ &= -\log(\sup_{i\in\mathbb{N}} m(a_i))(\sup_{k\in\mathbb{N}} \frac{\sup_{j\in\mathbb{N}} m(b_j \odot c_k)}{diam\ \mathscr{C}}) \\ &= -\log \sup_{i\in\mathbb{N}} m(a_i) - \log \sup_{k\in\mathbb{N}} \frac{diam(\mathscr{B}\nabla c_k)}{diam\ \mathscr{C}}. \end{aligned}$$

□

4 Entropy of $m-$preserving transformations

First we give some definitions.

Definition 4.1. Let G be a non-empty subset of F. We say G is $m-$set if there exists $k \in [0,1]$ such that $m(a) = k$, for any $a \in G$. In this case we define $m(G) = k$.

Definition 4.2. Let G_1 and G_2 be two non-empty subsets of F. We define,

$$G_1 \oplus G_2 = \{a_1 \oplus a_2 : a_1 \in G_1, a_2 \in G_2\},$$

$$G_1 \odot G_2 = \{a_1 \odot a_2 : a_1 \in G_1, a_2 \in G_2\}.$$

Remark 4.3. If G_1 and G_2 are $m-$sets, then $G_1 \oplus G_2$ is also an $m-$set.

Definition 4.4. A function $u : F \longrightarrow F$ is called $m-$preserving transformation if

(i) $u^{-1}(a)$ is an m-set with $k = m(a)$, for any $a \in F$;

(ii) $u^{-1}(a \oplus b)$ is an $m-$set and

$$m(u^{-1}(a \oplus b)) = m(u^{-1}(a) \oplus u^{-1}(b)),$$

for any $a, b \in F$;

(iii) $u^{-1}(a \odot b)$ and $u^{-1}(a) \odot u^{-1}(b)$ are $m-$sets and

$$m(u^{-1}(a \odot b)) = m(u^{-1}(a) \odot u^{-1}(b)),$$

for any $a, b \in F$.

Remark 4.5. It is easy to see that for an m-preserving transformation u, and for any $a \in F$ and $n \in \mathbb{N}$, we have

(i) $m(u^{-n}(a)) = m(a)$;

(ii) $m(u^{-n}(a \odot b)) = m(u^{-n}(a) \odot u^{-n}(b))$.

Definition 4.6. Let $u : F \longrightarrow F$ be an m-preserving transformation, and $\mathscr{A} = \{a\}_{i \in \mathbb{N}}$ be a countable partition in F. The inverse image of $\mathscr{A}$ by u is the set $u^{-1}\mathscr{A}$ containing exactly one element b_i of $u^{-1}(a_i)$, for any $a_i \in \mathscr{A}$.

Proposition 4.7. *The inverse image $u^{-1}\mathscr{A}$ is a countable partition in F, for any countable partition $\mathscr{A}$ in F, and any m-preserving transformation u. In addition,*

$$h(u^{-1}\mathscr{A}) = H(\mathscr{A}).$$

Proof. Let $u^{-1}\mathscr{A} = \{b_i \in F : u(b_i) = a_i, a_i \in \mathscr{A}\}$ such that, $(u^{-1}\mathscr{A} \setminus b_i) \cap (u^{-1}(a_i)) = \varnothing$. Therefore we have,

$$\sum_{i=1}^{\infty} m(b_i) = \sum_{i=1}^{\infty} m(u^{-1}(a_i)) = \sum_{i=1}^{\infty} m(a_i) = m(1).$$

On the other hand, since $u^{-1}(b)$ is an m-set for any $b \in F$, we have

$$\sum_{i=1}^{\infty} m(b_i \odot u^{-1}(b)) = \sum_{i=1}^{\infty} m(u^{-1}(a_i) \odot u^{-1}(b)) = \sum_{i=1}^{\infty} m(a_i \odot b) = m(b).$$

Thus $u^{-1}\mathscr{A}$ is a countable partition in F.

Now

$$H(u^{-1}\mathscr{A}) = -\log \sup_{i\in\mathbb{N}} m(b_i) = -\log \sup_{i\in\mathbb{N}} m(u^{-1}(a_i)) = -\log \sup_{i\in\mathbb{N}} m(a_i) = H(\mathscr{A}).$$

□

Corollary 4.8. *Let $\mathscr{B} = \{b_i : u(b_i) = a_i, i \in \mathbb{N}\}$ and $\mathscr{C} = \{c_i : u(c_i) = a_i, i \in \mathbb{N}\}$ be two inverse images of a countable partition $\mathscr{A} = \{a_i\}_{i\in\mathbb{N}}$ in F such that $b_i \neq c_i$, for some $i \in \mathbb{N}$. Then,*

$$H(\mathscr{B}) = H(\mathscr{C}).$$

Proof. It is clear by using previous proposition. □

Proposition 4.9. *Let $\mathscr{A}$ and $\mathscr{B}$ be two countable partitions in F, and u be an m-preserving transformation. If $\mathscr{A} \prec \mathscr{B}$, then $H(u^{-n}\mathscr{A}) \leq H(u^{-n}\mathscr{B})$.*

Proof. As $m(u^{-n}(a_i)) = m(a_i)$, for any $i \in \mathbb{N}$, we have

$$H(u^{-n}\mathscr{A}) = H(\mathscr{A}).$$

Similarly we have,

$$H(u^{-n}\mathscr{B}) = H(\mathscr{B}).$$

On the other hand, because $\mathscr{A} \prec \mathscr{B}$, we have $H(\mathscr{A}) \leq H(\mathscr{B})$. □

Proposition 4.10. *Let u be an $m-$preserving transformation, and $\mathscr{A}$ and $\mathscr{B}$ be two countable partitions in F. Then for any $n \in \mathbb{N}$,*

(i) $H(u^{-n}\mathscr{A}) = H(\mathscr{A})$;

(ii) $H(u^{-n}(\mathscr{A}\nabla\mathscr{B})) = H(u^{-n}\mathscr{A}\nabla u^{-n}\mathscr{B})$;

(iii) $H(u^{-n}\mathscr{A}|u^{-n}\mathscr{B}) = H(\mathscr{A}|\mathscr{B})$;

(iv) $H(u^{-n}\nabla_{i=1}^{k}\mathscr{A}) = H(\nabla_{i=1}^{k}u^{-n}\mathscr{A})$, *for any $k \in \mathbb{N}$.*

Proof.

(i) We show it by induction. For $n = 1$ it is obviously true. Suppose it's true for $n = k$. We have,

$$H(u^{-(k+1)}\mathscr{A}) = H(u^{-1}(u^{-k}\mathscr{A})) = H(u^{-k}\mathscr{A}) = H(\mathscr{A}).$$

(ii) It follows immediately from the definition.

(iii) For any $j \in \mathbb{N}$ we have $m(u^{-n}(b_j)) = m(b_j)$. It follows that,

$$diam(u^{-n}\mathscr{B}) = diam\mathscr{B}.$$

Similarly,

$$diam(u^{-n}(a_i)\nabla u^{-n}\mathscr{B}) = diam(a_i\nabla\mathscr{B}),$$

for any $i \in \mathbb{N}$.

(iv) By induction,

$$m(u^{-n}(a \odot \cdots \odot a)) = m(u^{-n}(a) \odot \cdots \odot u^{-n}(a)),$$

for any $n \in \mathbb{N}$.

□

Proposition 4.11. *Let u be an m-preserving transformation and $\mathscr{A}$ be a countable partition in F. Then for any $n \in \mathbb{N}$ we have,*

$$H(\nabla_{i=0}^{n-1}u^{-i}\mathscr{A}) = H(\mathscr{A}) + \sum_{j=1}^{n-1} H(\mathscr{A}|\nabla_{i=1}^{j}u^{-i}\mathscr{A}).$$

Proof. Let us prove it by induction. For $n = 1$ it is obviously clear. Now assume that this holds for $n = k$. So we have,

$$\begin{aligned} H(\nabla_{i=0}^{k}u^{-i}\mathscr{A}) &= H(\nabla_{i=1}^{k}u^{-i}\mathscr{A}\nabla\mathscr{A}) \\ &= H(\nabla_{i=1}^{k}u^{-i}\mathscr{A}) + H\mathscr{A}|\nabla_{i=1}^{k}u^{-i}\mathscr{A}). \end{aligned}$$

But

$$H(\nabla_{i=1}^{k}u^{-i}\mathscr{A}) = H(u^{-1}(\nabla_{i=0}^{k-1}u^{-i}\mathscr{A})) = (\nabla_{i=0}^{k-1}u^{-i}\mathscr{A}).$$

Then by using induction assumption, the equality is obtained. □

Lemma 4.12. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $a_{n+p} \leq a_n + a_p$, for any $n, p \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}.$$

For the proof see [7], chapter 4.

Proposition 4.13. *If u is an $m-$preserving transformation, and $\mathscr{A}$ is a countable partition in F, then $\lim_{n\to\infty} \frac{1}{n} H(\nabla_{i=0}^{n-1}u^{-i}\mathscr{A})$ exists.*

Proof. Let us set

$$a_n = H(\nabla_{i=0}^{n-1}u^{-i}\mathscr{A}).$$

Then,

$$a_{n+p} = H(\nabla_{i=0}^{n+p-1}u^{-i}\mathscr{A}) \leq H(\nabla_{i=0}^{n-1}u^{-i}\mathscr{A}) + H(\nabla_{i=n}^{n+p-1}u^{-i}\mathscr{A}).$$

But,

$$\begin{aligned} H(\nabla_{i=n}^{n+p-1} u^{-i}\mathscr{A}) &= -\log \sup_{j_i\in\mathbb{N}, 0\le i\le p-1} m(u^{-n}(a_{j_0}) \odot \cdots \odot u^{-(n+p-1)}(a_{j_{(p-1)}})) \\ &= -\log \sup_{j_i\in\mathbb{N}, 0\le i\le p-1} m(u^{-n}(a_{j_0} \odot \cdots \odot u^{-(p-1)}(a_{j_{p-1}}))) \\ &= -\log \sup_{j_i\in\mathbb{N}, 0\le i\le p-1} m(a_{j_0} \odot \cdots \odot u^{-(p-1)}(a_{j_{p-1}})) \\ &= H(\nabla_{i=0}^{p-1} u^{-i}\mathscr{A}). \end{aligned}$$

These imply that $a_{n+p} \le a_n + a_p$, for any $n, p \in \mathbb{N}$. □

Definition 4.14. Let u be an $m-$preserving transformation and $\mathscr{A}$ be a countable partition in F. Entropy of u with respect to $\mathscr{A}$ is defined by,

$$h(u,\mathscr{A}) = \lim_{n\to\infty} \frac{1}{n} H(\nabla_{i=0}^{n-1} u^{-i}\mathscr{A}).$$

Proposition 4.15. *Let $\mathscr{A}$ and $\mathscr{B}$ be two countable partitions in F, and u be an $m-$preserving transformation. Then we have,*

(i) $h(u,\mathscr{A}) \le H(\mathscr{A})$;

(ii) If $\mathscr{A} \prec \mathscr{B}$, *then* $h(u,\mathscr{A}) \le h(u,\mathscr{B})$;

(iii) $h(u, u^{-1}\mathscr{A}) = h(u,\mathscr{A})$;

(iv) $h(u, \mathscr{A}\nabla\mathscr{B}) \le h(u,\mathscr{A}) + h(u,\mathscr{B})$;

(v) $h(u, \nabla_{i=k}^{r} u^{-i}\mathscr{A}) = h(u,\mathscr{A})$, *for any* $k \le r, r \ge 0$;

(vi) $h(u, \nabla_{i=0}^{r} u^{-i}\mathscr{A}) = h(u,\mathscr{A})$;

(vii) If u is invertible and $k \ge 1$, then

$$h(u, \nabla_{i=-k}^{k} u^{-i}\mathscr{A}) = h(u,\mathscr{A}).$$

Proof.

(i) It is clear.

(ii) Because $\mathscr{A} \prec \mathscr{B}$, for $j = j_0$ there exists $l_0 \in \mathbb{N}$ such that,

$$m(b_{j_0}) \le m(a_{l_0}).$$

It follows that,

$$m(b_{j_0} \odot u^{-1}(b_{j_1})) \le m(a_l \odot u^{-1}(b_{j_1})).$$

Now for $j = j_1$ there exists $l_1 \in \mathbb{N}$ such that

$$m(b_{j_1}) \le m(a_{l_1}).$$

These imply that

$$m(b_{j_0} \odot u^{-1}(b_{j_1})) \leq m(a_{l_0} \odot u^{-1}(a_{l_1})).$$

Therefore using induction, we may find $l_0, ..., l_{n-1} \in \mathbb{N}$ for any $j_0, ..., j_{n-1} \in \mathbb{N}$ such that,

$$m(b_{j_0} \odot u^{-1}(b_{j_1}) \odot \cdots \odot u^{-(n-1)}(b_{j_{n-1}})) \leq m(a_{l_0} \odot u^{-1}(a_{l_1}) \odot \cdots \odot u^{-(n-1)}(a_{l_{n-1}})).$$

Note that if $m(b_j) \leq m(a_l)$, then $m(u^{-1}(b_j)) \leq m((u^{-1}(a_l))$, for any $n \in \mathbb{N}$.

(iii)

$$\begin{aligned} H(\nabla_{i=0}^{n-1} u^{-i}(u^{-1}\mathscr{A})) &= -\log \sup m(u^{-1}(a_{i_0} \odot u^{-1}(a_{i_1}) \odot \cdots \odot u^{-(n-1)}(a_{i_{n-1}}))) \\ &= -\log \sup m(a_{i_0} \odot u^{-1}(a_{i_1}) \odot \cdots \odot u^{-(n-1)}(a_{i_{n-1}})) \\ &= H(\nabla_{i=0}^{n-1} u^{-i}\mathscr{A}). \end{aligned}$$

(iv)

$$\begin{aligned} H(\nabla_{i=0}^{n-1} u^{-i}(\mathscr{A}\nabla\mathscr{B})) &= H(\nabla_{i=0}^{n-1}(u^{-i}\mathscr{A}\nabla u^{-i}\mathscr{B})) \\ &= H((\nabla_{i=0}^{n-1} u^{-i}(\mathscr{A})\nabla(\nabla_{i=0}^{n-1} u^{-i}\mathscr{B})) \\ &\leq H(\nabla_{i=0}^{n-1} u^{-i}(\mathscr{A}) + H(\nabla_{i=0}^{n-1} u^{-i}\mathscr{B}). \end{aligned}$$

(v)

$$\begin{aligned} h(u, \nabla_{i=k}^{r} u^{-i}\mathscr{A}) &= \lim_{n\to\infty} \frac{1}{n} H(\nabla_{j=0}^{n-1} u^{-j}(\nabla_{i=k}^{r} u^{-i}\mathscr{A})) \\ &= \lim_{n\to\infty} \frac{1}{n} H(\nabla_{i=0}^{n+r-k-1} u^{-i}\mathscr{A}) \\ &= \lim_{n\to\infty} \frac{n+r-k}{n} \frac{1}{n+r-k} H(\nabla_{i=0}^{n+r-k-1} u^{-i}\mathscr{A}) \\ &= h(u, \mathscr{A}). \end{aligned}$$

(vi) In (v) we set $k = 0$.

(vii) $h(u, \nabla_{i=-k}^{k} u^{-i}\mathscr{A}) = h(u, \nabla_{i=0}^{2k} u^{-i}\mathscr{A}) = h(u, \mathscr{A})$.

□

Definition 4.16. Let u be an $m-$preserving transformation. Entropy of u is defined by,

$$h(u) = \sup_{\mathscr{A}} h(u, \mathscr{A}),$$

where supremum is taken over all countable partitions in F.

Proposition 4.17. *If u is identity transformation, then $h(u) = 0$.*

Proof. By definition and using induction we see that $\nabla_{i=0}^{n-1}\mathscr{A} = \mathscr{A}$, for any $n \in \mathbb{N}$. Therefore,

$$h(id, \mathscr{A}) = \lim_{n\to\infty} \frac{1}{n} H(\nabla_{i=0}^{n-1}\mathscr{A}) = 0.$$

□

Corollary 4.18. *If u is an m-preserving transformation such that $u^k = id$, for some $k \in \mathbb{N}$, then $h(u^k) = 0$.*

Proof. This follows immediately from previous propositions. □

Proposition 4.19. *Let u be an $m-$preserving transformation. Then we have,*

(i) $h(u^k) = kh(u)$, for any $k > 0$;

(ii) If u is invertible, then $h(u^k) = |k|h(u)$, for any $k \in \mathbb{Z}$.

Proof.

(i) For any countable partition $\mathscr{A}$ in F we have,

$$\begin{aligned}
h(u^k, \nabla_{i=0}^{k-1} u^{-i}\mathscr{A}) &= \lim_{n\to\infty} \frac{1}{n} H(\nabla_{j=0}^{n-1}(u^k)^{-j}(\nabla_{i=0}^{k-1}u^{-i}\mathscr{A})) \\
&= \lim_{n\to\infty} \frac{1}{n} H(\nabla_{j=0}^{n-1}\nabla_{i=0}^{k-1}u^{-(kj+i)}\mathscr{A}) \\
&= \lim_{n\to\infty} \frac{1}{n} H(\nabla_{i=0}^{nk-1}u^{-i}\mathscr{A}) \\
&= \lim_{n\to\infty} \frac{nk}{n}\frac{1}{nk} H(\nabla_{i=0}^{nk-1}u^{-i}\mathscr{A}) \\
&= kh(u, \mathscr{A}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
kh(u) &= k \sup_{\mathscr{A}} h(u, \mathscr{A}) = \sup_{\mathscr{A}} h(u^k, \nabla_{i=0}^{k-1}u^{-i}\mathscr{A}) \\
&\leq \sup_{\mathscr{B}} h(u^k, \mathscr{B}) = h(u^k).
\end{aligned}$$

On the other hand, as

$$\mathscr{A} \prec \mathscr{A}\nabla u^{-1}A\nabla \cdots \nabla u^{-(k-1)}\mathscr{A},$$

we have,

$$h(u^k, \mathscr{A}) \leq h(u^k, \nabla_{i=0}^{k-1}u^{-i}A) = kh(u, \mathscr{A}).$$

(ii) We will show that $h(u^{-1}) = h(u)$.

$$\begin{aligned} h(u,\mathscr{A}) &= \lim_{n\to\infty}\frac{1}{n}H(\nabla_{i=0}^{n-1}u^{-i}\mathscr{A}) \\ &= \lim_{n\to\infty}\frac{1}{n}H(u^{-(n-1)}\nabla_{i=0}^{n-1}u^{-i}\mathscr{A}) \\ &= \lim_{n\to\infty}\frac{1}{n}H(\nabla_{i=0}^{n-1}u^{i-(n-1)}\mathscr{A}) \\ &= \lim_{n\to\infty}\frac{1}{n}H(\nabla_{i=0}^{n-1}u^{-i}\mathscr{A}) \\ &= h(u,\mathscr{A}). \end{aligned}$$

□

5 Generators of m-preserving transformations

Definition 5.1. Let $u : F \longrightarrow F$ be an $m-$preserving transformation. A countable partition $\mathscr{G}$ in F is said to be a generator of u if there exists $r \in \mathbb{N}$ such that,

$$\mathscr{A} \prec \nabla_{i=0}^{r}u^{-i}\mathscr{G},$$

for any countable partition $\mathscr{A}$ in F.

Proposition 5.2. *If $\mathscr{G}$ is a generator of u, then*

$$h(u,\mathscr{A}) \leq h(u,\mathscr{G}),$$

for any countable partition $\mathscr{A}$ in F.

Proof. $\mathscr{G}$ is a generator of u, so there exists $r > 0$ such that,

$$\mathscr{A} \prec \nabla_{i=0}^{r}u^{-i}\mathscr{G},$$

for any countable partition $\mathscr{A}$ in F. Therefore,

$$h(u,\mathscr{A}) \leq h(u,\nabla_{i=0}^{r}u^{-i}\mathscr{G}) = h(u,\mathscr{G}).$$

□

Proposition 5.3. *Let $\mathscr{G}$ be a generator of an m-preserving transformation u. Then,*

$$h(u) = h(u,\mathscr{G}).$$

Proof. Since $\mathscr{G}$ is a generator of u, we have $h(u,\mathscr{A}) \leq h(u,\mathscr{G})$. Thus,

$$\sup_{\mathscr{A}} h(u,\mathscr{A}) \leq h(u,\mathscr{G}).$$

And we know, $h(u,\mathscr{G}) \leq \sup_{\mathscr{A}} h(u,\mathscr{A})$. This completes the proof. □

6 Conclusions

In this paper, we have defined a specific algebraic structure and it's countable partition.Then we have proved some properties for the entropy of this countable partition, parallel to the properties of the classical entropy(see [7]). Also, we have represented the notion of $m-$preserving transformation. Finally, we have introduced a generator of a dynamical system and stated a version of Kolomogorov-Sinai theorem.

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