On L_1 convergence of certain cosine sums with twice quasi semi-convex coefficients

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Abstract. In this paper a criterion for L_1- convergence of a certain cosine sums with twice quasi semi-convex coefficients is obtained. Also a necessary and sufficient condition for L_1-convergence of the cosine series is deduced as a corollary.

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1 Introduction

It is well known that if a trigonometric series converges in L_1-metric to a function $f \in L_1$, then it is the Fourier series of the function f. Riesz [2] gave a counter example, showing that in a metric space L_1 the converse is not true. This motivated several authors to study L_1-convergence of the trigonometric series. During their investigations, some authors introduced modified trigonometric sums, as these sums approximate their limits better than the classical trigonometric series, in a sense that they converge in L_1-metric to the sum of the trigonometric series whereas the classical series do not. In this contest we introduce the following cosine sum defined by relation (1.1)

$$N_n^{(2)}(x) = \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^4 a_{j-2} - \Delta^4 a_{j-1})\cos kx + \frac{a_1(\cos x - 4)}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4},$$

and will show the L_1-convergence of this modified cosine sums with twice quasi semi-convex coefficients. In the sequel we will briefly describe the notations and definitions which are used throughout the paper.

In what follows we will denote by

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1.2}$$

with partial sums defined by

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx, \tag{1.3}$$

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and

$$g(x) = \lim_{n\to\infty} S_n(x). \tag{1.4}$$

In the sequel we will mention some results which are useful for the further work. Dirichlet's kernels are denoted by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\widetilde{D}_n(t) = \sum_{k=1}^{n} \cos kt$$

$$\overline{\overline{D}}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\overline{D}_n(t) = -\frac{1}{2}\cot\frac{t}{2} + \overline{\overline{D}}_n(t) = -\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

Definition 1.1. A sequence of scalars (a_n) is said to be semi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, (a_0 = 0), \tag{1.5}$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$.

Definition 1.2. A sequence of scalars (a_n) is said to be quasi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty, (a_0 = 0), \tag{1.6}$$

Definition 1.3. A sequence of scalars (a_n) is said to be twice quasi semi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^4 a_{n-1} - \Delta^4 a_n| < \infty, (a_0 = a_{-1} = 0), \tag{1.7}$$

where $\Delta^4 a_n = \Delta^3 a_n - \Delta^3 a_{n+1}$.

Definition 1.4. A sequence of scalars (a_n) is said to be twice quasi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^4 a_{n-1}| < \infty, (a_0 = a_{-1} = 0), \tag{1.8}$$

Remark 1.1. *If (a_n) is a twice quasi-convex null scalar sequence , then it is twice quasi semi-convex scalars sequence too.*

The L_1-convergence of cosine and sine sums was studied by several authors. Kolmogorov in [6], proved the following theorem:

Theorem 1.2. *If (a_n) is a quasi-convex null sequence, then for the L_1-convergence of the cosine series (1.2), it is necessary and sufficient that $\lim_{n\to\infty} a_n \cdot \log n = 0$.*

The case in which sequence (a_n) is convex, of this theorem was established by Young (see [11]).

Bala and Ram in [1] have proved that Theorem 1.2 holds true for cosine series with semi-convex null sequences in the following form:

Theorem 1.3. *If (a_n) is a semi-convex null sequence, then for the convergence of the cosine series (1.2) in the metric space L, it is necessary and sufficient that $a_{k-1}\log k = 0(1), k \to \infty$.*

Garret and Stanojevic in [4], have introduced modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta a_j)\cos kx. \tag{1.9}$$

The same authors (see [5]), Ram in [8] and Singh and Sharma in [9] studied the L_1-convergence of this cosine sum under different sets of conditions on the coefficients (a_n). Kumari and Ram in [10], introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right)\cos kx \tag{1.10}$$

and

$$G_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right)\sin kx, \tag{1.11}$$

and have studied their L_1-convergence under the condition that the coefficients (a_n) belong to different classes of sequences. Later one, Kulwinder in [7], introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1})\sin kx, \tag{1.12}$$

and have studied their L_1-convergence under the condition that the coefficients (a_n) are semi-convex null. In [3], was introduced the following modified cosine sums:

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

For this cosine sums was studied L_1-convergence under the condition that the coefficients (a_n) are quasi semi-convex null.

2 Results

The aim of this paper is to study the L_1-convergence of modified cosine sums given by relation (1.1), with twice quasi semi-convex coefficients and to give necessary and sufficient condition for L_1-convergence of the cosine series defined by relation (1.2).

Theorem 2.1. *Let (a_n) be a twice quasi semi-convex null sequence, then $N_n(x)$ converges to $g(x)$ in L_1 norm.*

Proof. We have

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cdot \cos kx = \frac{1}{\left(2\sin\frac{x}{2}\right)^4} \cdot \sum_{k=1}^{n} a_k \cdot \cos kx \cdot \left(2\sin\frac{x}{2}\right)^4$$

$$= \frac{1}{\left(2\sin\frac{x}{2}\right)^4} \cdot \sum_{k=1}^{n} a_k [\cos(k+2)x - 4\cos(k+1)x + 6\cos kx - 4\cos(k-1)x + \cos(k-2)x]$$

$$= \frac{1}{\left(2\sin\frac{x}{2}\right)^4} \cdot \sum_{k=1}^{n} (a_{k-2} - 4a_{k-1} + 6a_k - 4a_{k+1} + a_{k+2}) \cdot \cos kx - \frac{a_{-1}\cos x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{a_0 \cos 2x}{\left(2\sin\frac{x}{2}\right)^4} +$$

$$\frac{a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_n \cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4} + \frac{4a_0\cos x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{4a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{4a_1}{\left(2\sin\frac{x}{2}\right)^4} +$$

$$+ \frac{4a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_1\cos x}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4} - \frac{a_{n+1}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4} \Rightarrow$$

$$S_n(x) = \frac{1}{\left(2\sin\frac{x}{2}\right)^4} \cdot \sum_{k=1}^{n} \Delta^4 a_{k-2} \cos kx - \frac{a_{-1}\cos x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{a_0\cos 2x}{\left(2\sin\frac{x}{2}\right)^4} +$$

$$\frac{a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_n \cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4} + \frac{4a_0\cos x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{4a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{4a_1}{\left(2\sin\frac{x}{2}\right)^4} +$$

$$+ \frac{4a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_1\cos x}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4} - \frac{a_{n+1}\cos(n-1)x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}.$$

Applying Abel's transformation, we have

$$S_n(x) = \frac{1}{\left(2\sin\frac{x}{2}\right)^4} \cdot \sum_{k=1}^{n-1} (\Delta^4 a_{k-2} - \Delta^4 a_{k-1}) \widetilde{D}_k(x) - \frac{(\Delta^4 a_{n-2} - \Delta^4 a_{n-1}) \cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} +$$

$$\frac{a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{4a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{4a_1}{\left(2\sin\frac{x}{2}\right)^4} +$$

$$+ \frac{4a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_1\cos x}{\left(2\sin\frac{x}{2}\right)^4} + \frac{a_2}{\left(2\sin\frac{x}{2}\right)^4} - \frac{a_{n-1}\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4} - \frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}.$$

Since $\widetilde{D}_n(x)$ is uniformly bounded on every segment $[\epsilon, \pi - \epsilon]$, for every $\epsilon > 0$,

$$g(x)=\lim_{n\to\infty}S_n(x)=\frac{1}{\left(2\sin\frac{x}{2}\right)^4}\cdot\sum_{k=1}^{\infty}(\Delta^4a_{k-2}-\Delta^4a_{k-1})\widetilde{D}_k(x)+\frac{a_1(\cos x-4)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}$$

Also

$$N_n^{(2)}(x)=\frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^4a_{j-2}-\Delta^4a_{j-1})\cos kx+\frac{a_1(\cos x-4)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}$$

respectively

$$N_n^{(2)}(x)=\frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n}\Delta^4a_{k-2}\cos kx-\frac{\Delta^4a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_1(\cos x-4)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}.$$

Now applying Abel's transformation we get the following relation:

$$N_n^{(2)}(x)=\frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=1}^{n-1}(\Delta^4a_{k-2}-\Delta^4a_{k-1})\widetilde{D}_k(x)-\frac{\Delta^4a_{n-2}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}-\frac{\Delta^4a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}+$$
$$\frac{a_1(\cos x-4)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{a_2}{\left(2\sin\frac{x}{2}\right)^4}.$$

From the above relations we will have:

$$\begin{aligned}g(x)-N_n^{(2)}(x)= & \frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=n+1}^{\infty}(\Delta^4a_{k-2}-\Delta^4a_{k-1})\widetilde{D}_k(x)+\\ & +\frac{\Delta^4a_{n-2}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{\Delta^4a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4},\end{aligned}$$

whence

$$\begin{aligned}g(x)-N_n^{(2)}(x)= & \lim_{m\to\infty}\left(\frac{1}{\left(2\sin\frac{x}{2}\right)^4}\sum_{k=n+1}^{m}(\Delta^4a_{k-2}-\Delta^4a_{k-1})\widetilde{D}_k(x)\right)+\\ & +\frac{\Delta^4a_{n-2}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}+\frac{\Delta^4a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}.\end{aligned}$$

Thus, we have

$$\int_0^{\pi}|g(x)-N_n^{(2)}(x)|dx\to 0,$$

for $n\to\infty$, and definition 1.3. □

Corollary 2.2. *Let (a_n) be a twice quasi-convex null sequence, then $N_n^{(2)}(x)$ converges to $g(x)$ in L_1 norm.*

Proof. Proof of the corollary follows directly from Theorem 2.1 and Remark 1.1. □

Corollary 2.3. *If (a_n) is a twice quasi semi-convex null sequence of scalars, then the necessary and sufficient condition for L_1-convergence of the cosine series (1.2) is* $\lim_{n\to\infty} a_n \log n = 0$.

Proof. Let us start from this estimation:

$$||S_n(x)-g(x)||_{L_1} \le ||S_n(x)-N_n^{(2)}(x)||_{L_1}+||N_n^{(2)}(x)-g(x)||_{L_1} \le ||N_n^{(2)}(x)-g(x)||_{L_1}+$$

(2.1)

$$\left\|\frac{2\Delta^4 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4}\right\|+\left\|\frac{a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}\right\|+4\left\|\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}\right\|$$

On the other hand

$$\Delta^4 a_{n-1} = \sum_{k=n-1}^{\infty}(\Delta^4 a_k - \Delta^4 a_{k+1}) = \sum_{k=n-1}^{\infty}\frac{k}{k}(\Delta^4 a_k - \Delta^4 a_{k+1}) \le$$
$$\frac{1}{n-1}\sum_{k=n-1}^{\infty} k(\Delta^4 a_k - \Delta^4 a_{k+1}) = o\left(\frac{1}{n}\right).$$

Since

$$\int_0^{\pi}\frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} = O(n),$$

therefore

$$\Delta^4 a_{n-1}\cdot\int_0^{\pi}\frac{\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^4} = o(1).$$

For the rest of the expression (2.1) we have this estimation:

$$\int_0^{\pi}\left|\frac{a_n\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+2}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}\right|dx \le C_1\int_0^{\pi} a_n\left|\frac{\cos(n+2)x}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\cos nx}{\left(2\sin\frac{x}{2}\right)^2}\right|dx \le$$
$$C_1\cdot C_2\int_0^{\pi} a_n\left|\widetilde{D}_n(x)-\frac{1}{2}\right|dx \sim C_1\cdot C_2(a_n\log n).$$

In similar way we can estimate this expressions:

$$\int_0^{\pi}\left|\frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^4}-\frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^4}\right|dx \sim C_3(a_n\log n),$$

where C_1, C_2 and C_3 are constants. From Theorem 2.1 it follows that

$$||N_n^{(2)}(x)-g(x)|| = o(1), n\to\infty.$$

Finally we get this estimation

$$\lim_{n\to\infty}\int_0^{\pi}|g(x)-S_n(x)| = o(1),$$

if and only if

$$\lim_{n\to\infty} a_n \log n = 0,$$

which proves the corollary. □

Corollary 2.4. *If* (a_n) *is a twice quasi-convex null sequence of scalars, then the necessary and sufficient condition for* L_1*-convergence of the cosine series (1.2) is* $\lim_{n\to\infty} a_n \log n = 0$.

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