Applications of resultants in the spectral m-root framework

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Abstract. The $Z-$eigenvalues, $E-$eigenvalues and the corresponding eigenvectors for the Berwald-Moor associated multilinear form in the case $m = n = 3$, are computed by means of applying the method of resultants. The complexity of the algorithm and further developments are discussed.

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1 Introduction

In recent years, the theory of resultants has flourished, and its bases as stand-alone branch of mathematics have widely developed ([6], [7], [8], [10], [18], etc). At the same time, much attention has been payed to the spectral theory of supersymmetric tensors, whose numerous applications exhibit unexpectedly promising specific solutions ([14], [15], [16], [17], etc). The present paper applies the theory of resultants to the supersymmetric tensors which naturally emerges from the recently proposed 3-dimensional Berwald-Moor model for relativity theory ([11], [12], [5], etc), aiming to determine the main associated spectral related objects: the $Z-$, $H-$, $E-$eigenvalues, and the corresponding eigenvectors. It should be emphasized, that though the highly tedious straightforward computation technique can be successfully improved by software which leads to the desired goal ([2], [1]), the resultant theory is a useful aid which might significantly accelerate the employed spectral algorithms.

2 The resultant method

Generally, within the resultant theory, one may consider the problem of solving the system

$$\{\ F(x)|_{G(x)=0} \ = root\left(\hat{\mathcal{R}}\left[\left\{\begin{array}{l} C-F(x)=0 \\ G(x)=0 \end{array}\right\}, \{x\}\right], C\right). \tag{2.1}$$

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Further, the system (2.1) can be re-written as

$$\hat{\mathcal{R}}\left[\left\{\begin{array}{l} C-F(x)=0 \\ G(x)=0 \end{array}\right\},\{x\}\right]\Bigg|_{F,G\in Pol} = \left\{\mathcal{R}\left[\left\{\begin{array}{l} \mathcal{F}(C;h,x) \\ \mathcal{G}(h,x) \end{array}\right\},\{h,x\}\right]=0\right\}, \tag{2.2}$$

where

- $\hat{\mathcal{R}}[f(x)=0,\{x\}]$ is the compatibility condition of the system of equations $\{f(x)=0\}$;
- $\mathcal{R}\{f(x)\}$ is the resultant of the system of equations $\{f(x)=0\}$ relative to the variables $\{x\}$;
- $F(x)\in Pol(x^1,\dots x^n)$, $G(x)=(G^1(x),\dots G^k(x))$, $G^i(x)\in Pol(x^1,\dots x^n)$; $\mathcal{F}$ and $\mathcal{G}$ are practically determined by F and G, respectively;
- $root(P(C),C)$ is the set of roots of the algebraic equation $P(C)=0$ relative to C.

There exist several functions $\mathcal{F}$ and $\mathcal{G}$ which satisfy the condition (2.2). Depending on their choice, the complexity of the algorithm which determines $R\left[\left\{\begin{array}{l} \mathcal{F}(C;h,x) \\ \mathcal{G}(h,x) \end{array}\right\},\{h,x\}\right]$, strongly differs ([6], [7], [8]).

3 Eigenvalues and eigenvectors of multilinear supersymmetric tensors

We apply the stated technique to find the eigenvalues and eigenvectors of real supersymmetric tensors. It is known that there exist three type of eigenvalues - having corresponding associated eigenvectors, defined as follows ([14, 17]).

Definition 3.1. Consider a supersymmetric tensor $T\in\mathcal{T}^0_m(\mathbb{R}^n)$ of order m on $\mathbb{R}^n$. Then

a) We say that $\lambda\in\mathbb{R}$ is an $(Z-)$eigenvalue and a vector $y\in\mathcal{T}^1_0(\mathbb{R}^n)\equiv\mathbb{R}^n$ is an associated $(Z-)$eigenvector, if they satisfy the system:

$$Ty^{m-1}=\lambda y;\quad g(y,y)=1, \tag{3.1}$$

where we have considered the transvection

$$Ty^{m-1}=C^1_1C^2_2\dots C^{m-1}_{m-1}(T\otimes\underbrace{y\otimes\cdots\otimes y}_{m-1\text{ times}}),$$

C^i_j is the transvection operator on the corresponding indices and $y=y^ie_i$ is the Liouville vector field on the flat manifold $\mathbb{R}^n$, considered at some point $x\in\mathbb{R}^n$, and $g=\delta_{ij}dx^i\otimes dx^j$ is the flat Euclidean metric on $\mathbb{R}^n$ which implicitly raises/lowers the tensor indices. For the complex variant, λ and y are simply called *eigenvalue* and *eigenvector*, respectively.

b) We say that $\lambda \in \mathbb{R}$ is an $(H-)$eigenvalue and a vector $y \in \mathbb{R}^n$ is an associated $(H-)$eigenvector, if they satisfy the homogeneous polynomial system of order $m-1$:

$$(Ty^{m-1})_k = \lambda (y_k)^{m-1}. \tag{3.2}$$

For the complex variant, λ and y are simply called *(E-)eigenvalue* and *(E-)eigenvector*, respectively.

Regarding the existence of eigenvalues/eigenvectors, the following result holds true:

Theorem 3.1. *$H-$eigenvalues and $Z-$eigenvalues always exist for even supersymmetric tensors. A supersymmetric tensor T is positive definite/semi-definite iff all its $H-$ (or $Z-$) eigenvalues are positive/non-negative.*

The $Z-$, $H-$ and $E-$eigenvalues and the corresponding eigenvectors for the multilinear tensor (4.1) have been completely determined for the 4-dimensional case ([2], [1]). In the following, we show that employing resultants theory might significantly improve the computational task, and we illustrate this in the Berwald-Moor $\mathcal{H}_3$ particular case.

4 Applications of resultants theory - the $\mathcal{H}_3$ case

In this section, we shall apply the resultant theory for finding the eigenvalues and eigenvectors for an $m-$root type supersymmetric tensor with applications in Special Relativity, namely the $(0,3)$ Berwald-Moor tensor ([11, 12]):

$$A_{ijk} = \begin{cases} \dfrac{1}{3!}\,, & \text{for } \{i,j,k\} = \{1,2,3\} \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

associated to the $Berwald-Moor$ pseudonorm $F_{H_3} = \sqrt[3]{|y_1 y_2 y_3|}$;

These supersymmetric tensor provides a natural alternative model in SRT, which extends the 3-dimensional classical Minkowski models. The emerging new geometric framework is tightly related to the hypercomplex polynumbers theory and their applications. This leads both to the enhancement of the algebraic subjacent theory due to the geometrical viewpoint, and to the possibility of illustrating basic non-trivial and non-evident objects of the Berwald-Moor type approach by means of the relatively simple objects, such as the polynumbers ([11, 12, 5]).

4.1 Application: the $Z-$spectrum

Applying the method for super-symmetric polynomials $S(y)$, with $\deg(S) = k$, the $Z-$spectral problem leads to the system in the unknowns λ and $y = (y_1, \ldots, y_n)$,

$$\begin{cases} \partial_i S - \lambda y_i = 0 \\ \displaystyle\sum_{i=1}^{n} (y_i)^2 - 1 = 0, \end{cases} \tag{4.2}$$

where we denote $S := S(s_a)$, $s_a = \sum_{i=1}^{n}(y_i)^a$, and $\sigma_a = \sum_{i_1<\cdots<i_a} y_{i_1}\cdot\dots\cdot y_{i_a}$. For solving the system (4.2) we shall first focus on the adjacent problem of finding the quantities $p_a = \sigma_a|_{(4.2)}$. Using p_a, we further obtain the system $\{\sigma_a(y) - p_a = 0\}$, whence we infer that y_i are solutions of the algebraic equation

$$t^n - p_1 t^{n-1} + p_2 t^{n-2} + \cdots + (-1)^n p_n = 0, \tag{4.3}$$

which follows from the Viéte theorem.

This step practically allows to split the set of unknowns from (2.2). We subsequently obtain

$$root\left(\hat{\mathcal{R}}\left[\left\{\begin{array}{l} C-\sigma_a(y)=0\\ \partial_i S - \lambda y_i = 0\\ \sum_{i=1}^{n}(y_i)^2 - 1 = 0\end{array}\right\}, \{\lambda, y\}\right], C\right) = p_a \Leftrightarrow \hat{\mathcal{R}}\left[\left\{\begin{array}{l} C-\sigma_a(y)=0\\ \partial_i S - \lambda y_i = 0\\ \sum_{i=1}^{n}(y_i)^2 - 1 = 0\end{array}\right\}, \{\lambda, y\}\right] =$$
$$= \left(\mathcal{R}\left[\left\{\begin{array}{l} C\cdot h^n - h^{n-a}\cdot\sigma_a(y)\\ h^{n-k+1}\cdot\partial_i S - h^{n-2}\lambda y_i\\ h^{n-2}\sum_{i=1}^{n}(y_i)^2 - h^n\end{array}\right\}, \{\lambda, h, y\}\right] = 0\right), \tag{4.4}$$

where $\hat{\mathcal{R}}$ is the compatibility condition, and $\mathcal{R}$ is the resultant.

4.2 Application: the $H-$ and $E-$spectrum

For applying the resultant technique towards solving the $H-$ and $E-$spectral problems, we consider the case when $S = S(s_a)$, $s_a = \sum_{i=1}^{n}(y_i)^a$, with $\deg(S) = k$, and $m \le k \le n$ and we aim to solve the homogeneous system

$$\{\partial_i S - \lambda(y_i)^{m-1} = 0\}. \tag{4.5}$$

In this case we obtain the similar intermediate problem

$$\begin{array}{l} root\left(\hat{\mathcal{R}}\left[\left\{\begin{array}{l} C-\sigma_a(y)=0\\ \partial_i S - \lambda(y_i)^{m-1} = 0\end{array}\right\}, \{\lambda, y\}\right], C\right) = p_a \Leftrightarrow \\ \hat{\mathcal{R}}\left[\left\{\begin{array}{l} C-\sigma_a(y)=0\\ \partial_i S - \lambda(y_i)^{m-1} = 0\end{array}\right\}, \{\lambda, y\}\right] = \\ \left(\mathcal{R}\left[\left\{\begin{array}{l} C\cdot h^n - h^{n-a}\cdot\sigma_a(y)\\ h^{n-k+1}\cdot\partial_i S - h^{n-m+1}\lambda(y_i)^{m-1}\\ h^{n-2}\sum_{i=1}^{n}(y_i)^2 - h^n\end{array}\right\}, \{\lambda, h, y\}\right] = 0\right),\end{array} \tag{4.6}$$

where $\hat{\mathcal{R}}$ is the compatibility condition, and $\mathcal{R}$ is the resultant.

For applying the general theory to the case (4.1), we first note that in the first case, we have

$$Ty^3 = A_1\sigma_1^3 + A_2\sigma_1\sigma_2 + A_3\sigma_3,$$

where $Ty^3 = S_3$ and

$$\{\sigma_1 = y_1 + y_2 + y_3, \sigma_2 = y_1y_2 + y_2y_3 + y_3y_1, \sigma_3 = y_1y_2y_3\}$$

is the Gröbner basis of the space of 3-rd order polynomials. We note that for $A_1 = A_2 = 0, A_3 = 1$ we obtain the first case H_3. In this case we solve the equations

$$\{(Ty^2)_k - \lambda(y_k)^2 = 0\}. \tag{4.7}$$

We solve the problem of finding the conditions for which the system (4.7) has non-degenerate solutions (non-zero roots):

Theorem 4.1. *The system of homogeneous equations of order r $\{F_i(x) = 0\}$ has non-degenerate solutions iff $\mathcal{R}\{F_i(x)\} = 0$, where $\mathcal{R}$ is the resultant [6].*

Proof. In our case we compute $\mathcal{R}\{(Ty^2)_k - \lambda(y_k)^2\}$:

$$(Ty^2)_k - \lambda(y_k)^2 = \partial_k(S_3) - \lambda\partial_k(\frac{1}{3}s_3) = \partial_k(S_3 - \frac{1}{3}\lambda s_3),$$

where $\partial_k = \frac{\partial}{\partial y_k}$, $s_p = \sum_{i=1}^{3}(y_i)^p, p \in \{1, 2, 3\}$. Like $\{\sigma_p\}$, the family $\{s_p\}$ is a basis for the space of symmetric polynomials, and they are related via:

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - 2\sigma_2, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

Hence, we infer

$$\begin{aligned}\bar{S}_3 &= S_3 - \tfrac{1}{3}\lambda s_3 = S - \tfrac{1}{3}\lambda(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3) = \\ &= (A_1 - \tfrac{1}{3}\lambda)\sigma_1^3 + (A_1 + \lambda)\sigma_1\sigma_2 + (A_3 - \lambda)\sigma_3 = \bar{A}_1\sigma_1^3 + \bar{A}_2\sigma_1\sigma_3 + \bar{A}_3\sigma_3,\end{aligned}$$

where

$$\bar{A}_1 = A_1 - \frac{1}{3}\lambda, \quad \bar{A}_2 = A_2 + \lambda, \quad \bar{A}_3 = A_3 - \lambda.$$

Then the system (4.7) gets the form

$$\{\partial_k\bar{S}_3 = 0\}. \tag{4.8}$$

The resultant of the system (4.8) is a particular case of the basic function considered in the paper [13], and the conditions $\mathcal{R}\{\partial_i\bar{S}_3\} = 0$ are equivalent to any of the three conditions

$$\begin{cases} \bar{A}_3 = 0 \\ (2\bar{A}_3 - 3(\bar{A}_2 + \bar{A}_3))^3 - 9((\bar{A}_2 + \bar{A}_3)(2\bar{A}_3 - 3(\bar{A}_2 + \bar{A}_3)) - \\ \quad -4(\bar{A}_3(6\bar{A}_1\bar{A}_3 + \bar{A}_2\bar{A}_3 - \bar{A}_2^2)) = 0 \\ (2\bar{A}_3 - 3(\bar{A}_2 + \bar{A}_3))^3 - ((\bar{A}_2 + \bar{A}_3)(2\bar{A}_3 - 3(\bar{A}_2 + \bar{A}_3)) - \\ \quad -4(\bar{A}_3(6\bar{A}_1\bar{A}_3 + \bar{A}_2\bar{A}_3 - \bar{A}_2^2)) = 0. \end{cases} \tag{4.9}$$

If one of the three before mentioned conditions holds true, then the system (4.8) admits a one-parametric family of solutions. In the case when all the three conditions are fulfilled, then $\bar{A}_1 = \bar{A}_2 = \bar{A}_3 = 0$ and the system (4.8) is satisfied for any $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ (the 3-parametric family of solutions $\{y_1 = \tau_1, y_2 = \tau_2, y_3 = \tau_3\}$).

While solving the system $\{\partial_k \bar{S}_3 = 0\}$, we shall examine only the case when just one of the conditions of (4.9). In this case we infer the system of equations

$$\partial_k \bar{S}_3 = \partial_k \sigma_\alpha \cdot \bar{\partial}_\alpha S_3,$$

where $\bar{\partial}_\alpha = \frac{\partial}{\partial \sigma_\alpha}$ and, in the equation, we assume summation by α, and

$$\begin{cases} \partial_i \sigma_1 = 1 \\ \partial_i \sigma_2 = \sigma_1 - y_i \\ \partial_i \sigma_3 = \sigma_2 - \sigma_1 y_i + (y_i)^2 \end{cases}, \quad \begin{cases} \bar{\partial}_1 \bar{S}_3 = 3\bar{A}_1 \sigma_1^2 + \bar{A}_2 \sigma_2 \\ \bar{\partial}_2 \bar{S}_3 = \bar{A}_2 \sigma_1 \\ \bar{\partial}_3 \bar{S}_3 = \bar{A}_3. \end{cases}$$

Hence

$$\begin{aligned} \partial_i \bar{S}_3 + \bar{\partial}_1 \bar{S}_3 + (\sigma_1 - y_i)\bar{\partial}_2 \bar{S}_3 + (\sigma_2 - \sigma_1 y_i + (y_i)^2)\bar{\partial}_3 \bar{S}_3 = \\ = (\bar{\partial}_3 \bar{S}_3)(y_i)^2 + (-\bar{\partial}_2 \bar{S}_3 - \sigma_1 \bar{\partial}_3 \bar{S}_3) y_i + (\bar{\partial}_1 \bar{S}_3 + \sigma_1 \bar{\partial}_2 \bar{S}_3 + \sigma_2 \bar{\partial}_3 \bar{S}_3) = \\ = \bar{A}_3 (y_i)^2 + [-(\bar{A}_2 + \bar{A}_3)\sigma_1] y_i + [(3\bar{A}_1 + \bar{A}_2)\sigma_1^2 + (\bar{A}_2 + \bar{A}_3)\sigma_2] = 0. \end{aligned}$$

We shall first consider the case when the first relation in (4.9) is satisfied. From $\bar{A}_3 = 0$, after several computations in which we choose as natural parameter $\sigma = \sigma_1$, we find the following solution:

$$y_1 = \frac{1}{3}\sigma, \quad y_2 = \frac{1}{3}\sigma, \quad y_3 = \frac{1}{3}\sigma. \tag{4.10}$$

We examine further the cases when the last two relations in (4.9), are satisfied. In these cases we assume $\bar{A}_3 \neq 0$. For solving our system, we use the following transformation ([13, formula (5)]):

$$F_i = \frac{1}{\bar{A}_3} \partial_i \bar{S}_3 + \frac{\bar{A}_2 + \bar{A}_3}{\bar{A}_3 (2\bar{A}_3 - 3(\bar{A}_2 + \bar{A}_3))} \sum_{i=1}^{3} \partial_i \bar{S}_3.$$

We note that $2\bar{A}_3 - 3(\bar{A}_2 + \bar{A}_3) \neq 0$ (in the contrary case, the last two relations in (4.9) are simultaneously satisfied, fact which leads to the solution (4.10)), and hence the considered transformation is non-degenerate, since its determinant is $\dfrac{2}{\bar{A}_3^2 (2\bar{A}_3 - 3(\bar{A}_2 + \bar{A}_3))} \neq 0$. Then we obtain the following system, which is equivalent to the original one:

$$(y_i)^2 + 2A\sigma_1 y_i + B\sigma_1^2 = 0,$$

where

$$\begin{cases} A = -\dfrac{\bar{A}_2 + \bar{A}_3}{2\bar{A}_3}, \\ B = \dfrac{6\bar{A}_1 \bar{A}_3 + 2\bar{A}_2 \bar{A}_3 - \bar{A}_2^2}{\bar{A}_3 (2\bar{A}_3 - 3(\bar{A}_2 + \bar{A}_3))}. \end{cases}$$

Further, choosing as natural parameter $\sigma = \sigma_1$, we finally infer

$$y_1 = p_1\sigma, \ \ y_2 = p_2\sigma, \ \ y_3 = p_3\sigma, \tag{4.11}$$

with $p_1, p_2, p_3 \in \{\mu_1, \mu_2\}$, with

$$\begin{cases} \mu_1 = -A + (A^2 - B)^{1/2} \\ \mu_2 = -A - (A^2 - B)^{1/2}. \end{cases}$$

Further, we shall examine the case of the system

$$\begin{cases} (Ty^{n-1})_k - \lambda y_k = 0 \\ (y_1)^2 + \cdots + (y_n)^2 = 1, \end{cases}$$

for $Ty^3 = S_3 = A_1\sigma_1^3 + A_2\sigma_1\sigma_2 + A_3\sigma_3$. □

We shall further use the basis $\{s_1, s_2, s_3\}$, $s_a = \sum_{i=1}^{3}(y_i)^a$ of the space of 3-rd order symmetric polynomials. We note that $S_3 = A_1 s_1^3 + A_2 s_1 s_2 + A_3 s_3$, where A_i are certain constants. We aim to solve with respect to $\{\lambda, y_1, y_2, y_3\}$ the system

$$\begin{cases} \partial_i S_3 - \lambda y_i = 0, \ i \in \{1, 2, 3\} \\ s_2 = 1. \end{cases} \tag{4.12}$$

While solving (4.12), we shall impose on A_1, A_2, A_3 no additional constrains, hence we shall examine the most general case. The system (4.12) is equivalent to

$$\begin{cases} \displaystyle\sum_{i=1}^{3}(y_i)^m \cdot (\partial_i S_3 - \lambda y_i) = 0, \ m \in \{0, 1, 2\} \\ s_2 = 1. \end{cases} \tag{4.13}$$

The (4.13) leads to the system whose unknowns are $\{\lambda, s_1, s_2, s_3\}$:

$$\begin{cases} \displaystyle\sum_{a=1}^{3} a \cdot s_{a+m-1}\bar{\partial}_a \left(S_3 - \frac{1}{2}\lambda s_2\right) = 0, \ m \in \{0, 1, 2\} \\ s_2 = 1. \end{cases} \tag{4.14}$$

Assuming that the solutions of the system (4.14) are found, then the solutions of (4.13) can be determined as follows. Let $\{\lambda(A_k); s_1(A_k), s_2(A_k), s_3(A_k)\}$ be the solutions off (4.14). Then each solution of (4.14) will lead to a class of solutions of (4.13), denoted by $\{\lambda(A_k);\, p_1(A_i),\, p_2(A_i),\, p_3(A_i)\}$, where (p_1, p_2, p_3) are the solutions of the equation

$$p^3 - \sigma_1 p^2 + \sigma_2 p - \sigma_3 = 0, \tag{4.15}$$

with $\{\sigma_a\}$ functions of s_a, as follows:

$$\begin{cases} \sigma_1 = s_1 \\ \sigma_2 = \frac{1}{2}s_1^2 - \frac{1}{2}s_2 \\ \sigma_3 = \frac{1}{6}s_1^3 - \frac{1}{2}s_1 s_2 + \frac{1}{3}. \end{cases}$$

It is important to stress that the solutions of (4.15) are expressed in terms of radicals, and, for briefness, we shall not include their explicit expressions. We solve in the following the system (4.14). Since in our case the basis of symmetric polynomials is $\{s_1, s_2, s_3\}$, then s_a for $a \geq 4$ will functionally depend on s_1, s_2, s_3. In particular, $s_4 = \frac{1}{6}s_1^4 - s_1^2 s_2 + \frac{4}{3}s_1 s_3 + \frac{1}{2}s_2^2$. Hence,

$$\bar{\bar{\partial}}_m \equiv \sum_{a=1}^{3} a \cdot s_{a+m-1}\bar{\partial}_a, \quad m \in \{0, 1, 2\}, s_0 \equiv 1$$

will have the following explicit form:

$$\begin{pmatrix} \bar{\bar{\partial}}_0 \\ \bar{\bar{\partial}}_1 \\ \bar{\bar{\partial}}_2 \end{pmatrix} = \begin{pmatrix} [1] & [2s_1] & [3s_2] \\ [s_1] & [2s_2] & [3s_3] \\ [s_2] & [2s_3] & [3s_4] \end{pmatrix} \begin{pmatrix} \bar{\partial}_1 \\ \bar{\partial}_2 \\ \bar{\partial}_3 \end{pmatrix}, \tag{4.16}$$

where $s_4 = \frac{1}{6}s_1^4 - s_1^2 s_2 + \frac{4}{3}s_1 s_3 + \frac{1}{2}s_2^2$. Having in view that

$$\begin{pmatrix} \bar{\partial}_1 \\ \bar{\partial}_2 \\ \bar{\partial}_3 \end{pmatrix} (s_3 - \frac{\lambda}{2}s_2) = \begin{pmatrix} 3A_1 s_1^2 + A_2 s_2 \\ A_2 s_1 - \frac{1}{2}\lambda \\ A_3 \end{pmatrix},$$

the equations (4.14) lead to the following equivalent system

$$\begin{cases} \begin{pmatrix} [1] & [2s_1] & [3s_2] \\ [s_1] & [2s_2] & [3s_3] \\ [s_2] & [2s_3] & [3s_4] \end{pmatrix} \begin{pmatrix} 3A_1 s_1^2 + A_2 s_2 \\ A_2 s_1 - \frac{1}{2}\lambda \\ A_3 \end{pmatrix} = 0 \\ s_4 = \frac{1}{6}s_1^4 - s_1^2 s_2 + \frac{4}{3}s_1 s_3 + \frac{1}{2}s_2^2 \\ s_2 = 1. \end{cases} \tag{4.17}$$

Simplifying, we infer the following $3 + 2$ equations

$$\begin{cases} \begin{pmatrix} [1] & [2s_1] & [3] \\ [s_1] & [2] & [3s_3] \\ [1] & [2s_3] & [3s_4] \end{pmatrix} \begin{pmatrix} 3A_1 s_1^2 + A_2 s_2 \\ A_2 s_1 - \frac{1}{2}\lambda \\ A_3 \end{pmatrix} = 0 \\ s_4 = \frac{1}{6}s_1^4 - s_1^2 + \frac{4}{3}s_1 s_3 + \frac{1}{2} \\ s_2 = 1. \end{cases} \tag{4.18}$$

Eliminating from the second equation of the system (4.18) the variables λ, s_3, s_4, we get the system of 4 equations:

$$\begin{cases} \lambda = \dfrac{1}{s_1}((3A_1 + 2A_2)s_1^2 + A_2 + 3A_3) \\ (s_1^2 - 1)[(18A_1^2 - 24A_1A_3 + 3A_3^2)(s_1^2)^2 + (12A_1A_2 + 36A_1A_3 - \\ \qquad -8A_2A_3 - 15A_3^2)s_1^2 + (2A_2^2 + 12A_2A_3 + 18A_3^2)] = 0 \\ s_2 = 1 \\ s_3 = \dfrac{1}{3A_3 s_1}(-3A_1 s_1^4 - A_2 s_1^2 + 3A_1 s_1^2 + A_2 + 3A_3). \end{cases} \tag{4.19}$$

The solutions of the second equation of the system (4.19), according to the rest of the equations of the system, form an infinite set $\{\lambda(A_k), s_1(A_k), s_2(A_k), s_3(A_k)\}$. It is essential to stress that the second equation of (4.19) has its solution expressed in terms of radicals, and for briefness, we shall not include its explicit form.

Hence we have obtained that, considering (4.15) and the second equation of (4.19), the solution of the original system (4.12) can be expressed in terms of radicals (is rational), and its exact form is given by (4.15) and (4.19). We note, that the system (4.19) admits, as well, solutions which do not satisfy the system (4.13). These roots correspond to the solutions of the second equation of (4.19), $s_1 = \pm 1$. By discarding them, we obtain the final set of solutions.

5 Conclusions

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