# On Nambu bracket representations of 3-algebras

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**Abstract.** We analyze irreducible representations of Nambu–Lie 3-algebras in terms of matrix operators subject to Nambu bracket relations. In general context we establish the criterium for existence of Hermitian representations. The case of Euclidean four-dimensional 3-algebra is considered in details. In this case it appears that this criterium is broken. We find that in spite of its Euclidean nature this algebra does not allow any Hermitian irreducible representation.

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## 1 Introduction

Recent proposals for the theory describing multiple M2 branes [3, 2, 1, 16] revived the general interest towards structures appearing in Nambu mechanics [24]. A particularly dramatic issue is related to Nambu–Lie algebras [17, 4, 18, 25, 14, 26, 19, 21, 22, 27, 6]. Already in [24] it was noted that there are only very restricted possibilities for certain types of a non-trivial Nambu–Lie 3-algebra. This subject was further developed by [28, 7, 12, 5]. Thus the only "compact", i.e. having positive scalar product finite dimensional algebra was found to be the four-dimensional algebra $\mathcal{A}_4$ with structure constants given by the totally anti-symmetric four-dimensional tensor[25, 14].

On the other hand, for the viability of the multiple M2-brane model we need an infinite family of 3-algebras parameterized by an integer parameter, which can be interpreted as the integer M2-brane charge.

In fact, giving up the Euclidean metric or finite dimension requirement makes it possible to construct various 3-algebras from Lie algebras by an uplift procedure. It is clear that all such 3-algebras could be either Lorentzian or infinite-dimensional [13, 15, 22].

In this work we analyze a construction of representations of Nambu–Lie 3-algebras in terms of matrix operators subject to Nambu commutation relations. The construction is based on the fact that the 3-algebra includes a family of underlying Lie algebra structures, which are obtained by fixing a 3-algebra element. Then, the 3-algebra representation decays into and therefore is composed of underlying Lie algebra representations. Let us note, that a Nambu bracket algebra, even closed, does not represent generally a *good* 3-algebra since Nambu bracket does not satisfy the Fundamental

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Identity (FI) automatically as does the Lie commutator. However, in the case in which the Nambu bracket algebra is homomorphic to a 3-algebra which satisfies the FI, so does the Nambu bracket algebra too.

The construction can be applied to either Lorentzian or infinite dimensional 3-algebras, but in this work we concentrate mainly on the case of four-dimensional Euclidean 3-algebra $\mathcal{A}_4$. It appears that, in spite of its Euclidean nature, this 3-algebra can not be represented in terms of sole Hermitian matrices. The uplift of a representation of a compact Lie algebra requires complex eigenvalues for the fixed element parameterizing the underlying Lie algebra. If this is a general property, it has a strong impact on possibility of construction of physical models based on 3-algebras. We analyze the situation in a more general setup to reach the same conclusion.

The plan of the paper is as follows. In the next section we discuss the matrix representations of a 3-algebra and describe the requirements they should satisfy. Next, we give a general construction of such a representation. After that we specify to algebra $\mathcal{A}_4$ for which we find the explicit form of representations. Then we analyze the two-dimensional representation in most general context, but find that this case too in fact reduces to the previous one.

# 2 Lie algebra induced representation

## 2.1 Objectives

Consider a Nambu–Lie 3-algebra defined as a linear span of generators $\{T_a\}$, $a = 1, \ldots, D$ subject to the following 3-bracket relation,

$$[T_a, T_b, T_c] = \mathrm{i}\, f_{abc}{}^{d} T_d, \tag{2.1}$$

where the real valued structure constants $f_{abc}{}^{d}$ are anti-symmetric in the lower indices and subject to FI,

$$f_{a_1 a_2 b a} f_{a_3 a_4 a_5 b} = f_{a_1 a_2 a_3 b} f_{b a_4 a_5 a} + f_{a_1 a_2 a_4 b} f_{a_3 b_2 a_5 a} + f_{a_1 a_2 a_5 b} f_{a_3 b_2 a_5 a}, \tag{2.2}$$

for all $a$. The indices are lowered and raised respectively by the 3-algebra analog of Killing metric (not necessarily Euclidean) $h_{ab}$ and its inverse. We suppose that the structure constants $f_{abc}{}^{d}$ are non-degenerate in the sense that for at least one vector $v^a$ there exist "dual structure constants" $\tilde{f}^{abc}{}_{f}$ such that,

$$v^f \tilde{f}^{abc}{}_{f} f_{abc}{}^{d} = v^d. \tag{2.3}$$

This is not a very constraining requirement, in any case any semi-simple Lie algebra satisfies to a much more general analogue of this having arbitrary $v$. In the case of totally anti-symmetric structure constants $f$, the dual constants can be chosen to be totally anti-symmetric as well.

Our aim is to construct a matrix representation $R(\mathcal{A})$ such that the Nambu-Lie bracket is mapped to the Nambu commutator [24],

$$[A, B, C]_N = ABC + BCA + CAB - BAC - ACB - CBA, \tag{2.4}$$

where $A, B, C \in R(\mathcal{A})$.

Let us note, that, as discussed in the introduction, unlike the Lie commutator, the Nambu bracket (2.4) does not satisfy the FI in general. However, when the original 3-algebra, which is represented through the Nambu commutator satisfies the FI, it is obvious that the homomorphic matrix algebra with Nambu commutator also satisfies this identity. One may dispute the opportunity of imposing FI (2.2), proposing its replacement with a more permissive condition which is, in particular, automatically compatible with the Nambu bracket structure (see e.g. [11, 10, 9, 8]). In spite of this point of view and motivated by the recent development in the multiple brane description, we insist to analyze the Nambu commutator representations of 3-algebras subject to FI (2.2).

## 2.2 General construction

As is known a Nambu-Lie $n$-algebra generates a family of lower degree Nambu-Lie algebras. In the case of 3-algebra one has a family of Lie brackets parameterized by an element $\vec{\xi} \in \mathcal{A}$,

$$[g, f]_{\vec{\xi}} \equiv [\vec{\xi}, g, f], \qquad \vec{\xi}, g, f \in \mathcal{A}. \tag{2.5}$$

As pointed in [28], the restricted bracket (2.5) is anti-symmetric in $g$ and $f$ and satisfies the Bianchi identity, therefore a Nambu–Lie algebra contains incorporated Lie algebra structures.

Let the fixed element be identified with $T_D$. Then the underlying (d−1)-dimensional Lie algebra is defined by the commutation relations,

$$[T_i, T_j]_{t_D} \equiv [T_D, T_i, T_j] = \mathrm{i}\, f_{ij}{}^k T_k, \tag{2.6}$$

where $f_{ij}{}^k = f_{Dij}{}^k$, $i, j, k = 1, \ldots, D-1$.

Since we seek a hermitian representation of 3-algebra, it is natural to assume, that the the matrix representing $T_D$ can be diagonalized within the underlying Lie algebra representation. Consider the subspace corresponding to the eigenvalue $t_D$. The Nambu–Lie commutator on such a space reduces to[1]

$$[T_i, T_j]_{t_D} = t_D [T_i, T_j], \tag{2.7}$$

where $[\cdot, \cdot]$ is the usual matrix commutator. In order to satisfy the Nambu–Lie commutation relations (2.6), the reduced representation on $t_D$ subspace should satisfy the following matrix commutation relations,

$$[T_i, T_j] = \mathrm{i}\, t_D^{-1} f_{ij}{}^k T_k. \tag{2.8}$$

Let us consider the situation when the underlying algebra is a semi-simple Lie algebra. Then the $t_D$-eigen-space is split into irreducible representations of the semi-simple algebra. The irreducible representations of the algebra (2.7) correspond to eigen-spaces of the quadratic Casimir operator,

$$C_2(T) = t_D^{-2} c_2(\tau), \tag{2.9}$$

[1] Note that here we abusively denote both algebra generators and the their representation as $T_i$.

where $c_2(\tau)$ is the quadratic Casimir of the $t_D$-independent algebra with normalized commutation relations,

$$[\tau_i, \tau_j] = \mathrm{i}\, f_{ij}{}^k \tau_k, \tag{2.10}$$

where $\tau_i = t_D T_i$.

Now let us pick up a particular irreducible representation and compute the following quantity restricted to this representation,

$$t_D \equiv T_D = -\mathrm{i}\ \tilde{f}^{ijk}{}_D [T_i, T_j, T_k]_N = 3t_D^{-3} \tilde{f}^{ijk}{}_D f_{jk}{}^l \tau_i \tau_l. \tag{2.11}$$

This implies that the value $t_D$ of the matrix $T_D$ should satisfy,

$$t_D^4 = \tilde{f}^{ijk}{}_D f_{jk}{}^l \tau_i \tau_l, \tag{2.12}$$

where the r.h.s. depends only on the details of the representation of the Lie algebra and should be related to the value of Casimir operator. So, the eigenvalue of $T_4$ is expressed as the fourth power root from the r.h.s of (2.12). It is clear, that if the r.h.s fails to be positive, the representation can not be realized in terms of Hermitian matrices.

In the case in which the r.h.s. of (2.12) contains beyond the Casimir also a Lie-algebra part, the consistency with Nambu–Lie commutation relation requires that this part should vanish. It is not clear how restrictive this condition is since not much is known on Nambu–Lie algebras in general, however in the case of $\mathcal{A}_4$ 3-algebra considered below this does not imply any additional constraints.

### Irreducibility

Constructing matrix representations of a Nambu–Lie algebra, one shall ask her/himself about the criteria of irreducibility of the representation. If we define an *irreducible representation* as one for which *the representation module has no invariant subspaces others than itself or zero element*, then if we restrict ourself to a subspace with definite value of $T_D$, the irreducibility of Nambu–Lie algebra representation follows directly from the irreducibility of the underlying Lie algebra representation.

It is worth noting that this definition also leads to a 3-algebra analog of *Schur's Lemma*. Indeed, any operator which is Nambu commuting with any two arbitrary elements of the 3-algebra should commute in the sense of Lie algebra commutation with all generators of the underlying Lie algebra and, therefore, be proportional to the identity operator on each irreducible representation of the latter. That is not all, however. The Nambu bracket involving *two* Lie algebra generators reads,

$$0 = [T_i, T_j, X] = [T_i, T_j]X = \mathrm{i}\, X f_{ijk} T_k \Rightarrow X = 0, \tag{2.13}$$

where we used the fact that $X$ is commuting with all generators $T_i$ following from the other Nambu bracket relations,

$$0 = [T_D, T_i, X]_N = [T_i, X]T_D + [X, T_D]T_i \Rightarrow [T_i, X] = 0, \tag{2.14}$$

for a irreducible underlying Lie algebra representation. Then the 3-algebra analogue of the Schur's lemma is somehow more restrictive:

**Lemma 1.** *There are no central elements in an irreducible representation of a Nambu Lie 3-algebra except the trivial one:* $X = 0$.

In particular this implies that there are no analogues of Casimir operators for 3-algebras.

## 2.3 Algebra $\mathcal{A}_4$

So far the only known nontrivial example of Euclidean finite-dimensional 3-algebra is the four dimensional 3-algebra $\mathcal{A}_4$, which is generated by the Nambu–Lie commutation relations,

$$[T_a, T_b, T_c] = \mathrm{i}\,\epsilon_{abcd} T_d, \tag{2.15}$$

where $\epsilon_{abcd}$ is the four-dimensional totally anti-symmetric tensor with $\epsilon_{1234} = +1$. Moreover, there is a strong evidence [28, 14, 25, 18] that it is the only non-trivial Euclidean 3-algebra in finite dimensions.

As $\mathcal{A}_4$ is invariant with respect to SO(4) rotations, all underlying Lie algebras are related by a SO(4) rotation and are isomorphic to su(2). Therefore, the choice of $T_4$ for the reduction of the Nambu–Lie bracket into a Lie algebra is a generic one. The underlying Lie algebra commutation relations are,

$$[T_i, T_j]_{T_4} = -\,\mathrm{i}\,\epsilon_{ijk} T_k, \quad i, j, k = 1, 2, 3, \tag{2.16}$$

where $\epsilon_{ijk}$ is the three-dimensional totally anti-symmetric tensor with $\epsilon_{123} = +1$.

On the other hand, the Nambu bracket commutation relations reduced to an eigen-space of $T_4$ transform to the following matrix commutator relations,

$$[T_i, T_j] = -\,\mathrm{i}\,t_4^{-1} \epsilon_{ijk} T_k. \tag{2.17}$$

Irreducible $(2j + 1)$-dimensional representations of the Lie algebra generated by (2.17) are parameterized by the half-integer spin $j$. The Casimir operator is given by,

$$T_i^2 = t_4^{-2} \tau_i^2 = t_4^{-2} j(j + 1). \tag{2.18}$$

Consideration of the leftover Nambu–Lie bracket $[T_1, T_2, T_3]$ gives the constraint on the value of $T_4$,

$$\begin{aligned} \mathrm{i}\,t_4 \equiv \mathrm{i}\,T_4 = [T_1, T_2, T_3]_N &= T_1[T_2, T_3] + T_2[T_3, T_1] + T_3[T_1, T_2] \\ &= -\,\mathrm{i}\,t_4^{-1}(T_1^2 + T_2^2 + T_3^2) = -\,\mathrm{i}\,t_4^{-3} j(j + 1), \end{aligned} \tag{2.19}$$

from which we have

$$t_4^4 = -j(j + 1). \tag{2.20}$$

As one can see it is not possible in this simple setup to represent the algebra $\mathcal{A}_4$ in terms of Hermitian matrices only. Let us note that this type of representations were first considered in [20] (see also [18]), in which the minus sign in front of $t_4^4$ was mishandled.

## 2.4 Nonconstant $T_4$ representation

One may be concerned that the impossibility to find a unitary representation of $\mathcal{A}_4$ is related to the fact that we considered only the representations with a constant value of $T_4$. This implies that $T_4$ commutes with other three generators $T_i$ on each irreducible space. One may wonder whether giving up the commutativity of $T_4$ may improve the situation.

For non-commuting $T_4$ the Nambu commutator (2.15) reduces to the following Lie bracket,

$$[A, B_{T_4} \equiv T_4[A, B] + [A, B]T_4 - AT_4B + BT_4A. \tag{2.21}$$

One can consider this re ]]ation as a $T_4$ dependent deformation of usual commutation relations. Therefore, a $d_j$-dimensional representation in terms of bracket (2.21) should be a deformation of an ordinary matrix representation of the underlying Lie algebra.

Consider the algebra $\mathcal{A}_4$. Here let us limit ourselves to the two-dimensional representation of the underlying algebra su(2). The deformed commutation relations in this case read,

$$\mu[\tau_i, \tau_j] + [\tau_i, \tau_j]\mu - \tau_i\mu\tau_j + \tau_j\mu\tau_i = \mathrm{i}\,\epsilon_{ijk}\tau_k, \tag{2.22}$$

where the generators $\tau_i$, $i = 1, 2, 3$ are 2×2-dimensional $\mu$-dependent matrices while $\mu \equiv T_4$, for notational convenience.

To find a "solution" for the commutation relations (2.22) let us expand everything in terms two-dimensional Pauli matrices,

$$\mu = \mu_0\mathbb{I} + \mu_\alpha\sigma_\alpha, \quad \tau_i = \tau_0\mathbb{I} + \tau_{i\alpha}\sigma_\alpha. \tag{2.23}$$

Substituting this expansion into (2.22) yields,

$$2\mu_0\epsilon_{\alpha\beta\gamma}\tau_{i\alpha}\tau_{j\beta}\sigma_\gamma + 6\mu_\sigma\epsilon_{\alpha\beta\sigma}\tau_{i\alpha}\tau_{j\beta} = \epsilon_{ijk}(\tau_{k\alpha}\sigma_\alpha + \tau_{k0}). \tag{2.24}$$

The equation (2.24) is satisfied iff each of the following equalities hold,

$$2\mu_0\epsilon_{\alpha\beta\gamma}\tau_{i\alpha}\tau_{j\beta} = \mathrm{i}\,\epsilon_{ijk}\tau_{k\gamma}, \quad 6\epsilon_{\alpha\beta\gamma}\tau_{i\alpha}\tau_{j\beta}\mu_\gamma = \epsilon_{ijk}\tau_{k0}. \tag{2.25}$$

The equalities (2.25) can be equivalently rewritten as,

$$[\hat{\tau}_i, \hat{\tau}_j] = \mathrm{i}\,\mu_0^{-1}\epsilon_{ijk}\hat{\tau}_k, \quad \hat{\tau}_i = \tau_{i\alpha}\sigma_\alpha/2 \tag{2.26}$$

and

$$\tau_{k0} = \tfrac{3}{2}\mu_0^{-1}\,\mathrm{tr}\,\hat{\mu}\hat{\tau}_k = 3\mu_0^{-1}\mu_\alpha\tau_{k\alpha}. \tag{2.27}$$

To complete the solution to the problem, let us note that for arbitrary matrix $\mu$ the solution for $\hat{\tau}_i$ satisfying (2.26) is given up to a unitary transformation in terms of Pauli matrices: $\hat{\tau}_i = \sigma_i/(2\mu_0)$ and $\tau_{k0} = \frac{3}{2}\mu_k/\mu_0^2$. Plugging these results into the leftover Nambu commutator we obtain the equality,

$$\mu = -\mathrm{i}\,[\tau_1, \tau_2, \tau_3] = -\mu_0^{-3}\mu_k\hat{\tau}_k - \tfrac{3}{4}\mu_0^{-3}, \tag{2.28}$$

from which we get,

$$\mu_0^4 = -3/4, \qquad \mu_k = 0, \tag{2.29}$$

i.e. the representation reduces to one considered in the previous subsection.

Taking the above results we may conjecture that complex values of $T_4$ are unavoidable in general.

# 3 Discussion

In this work we analyzed the construction of irreducible Nambu commutator representations of 3-algebras in terms of representations of underlying Lie algebras. It appears that irreducibility of the Lie algebra representation automatically leads to the irreducibility of the entire Nambu–Lie 3-algebra representation.

Applied to the case of four-dimensional 3-algebra $\mathcal{A}_4$ the construction results in representations, which necessarily require matrices with eigenvalues which are fourth roots of $-1$. This means that such representations can not be built entirely of hermitian matrices. This conclusion appears to be quite general.

Let us note that the Hermiticity of representations is required for physical applications. Thus for a non-Hermitian representation the kinetic term even in the case of Euclidean algebras fails to be positive definite.

One may try the following possibilities in order to overcome this difficulty. (i) Changing the sign of some brackets resulting in the change in the sign of some structure constants may solve the problem, but the resulting set of structure constants will fail to obey FI. (ii) One may hope that when considering Lorentzian 3-algebras obtained by the uplift procedure from the Lie algebra the sign flip can be compensated by the negative norm. (iii) Consider the situation when the fixed quantity is not an element of the algebra but e.g. a result of a bracket operation inherited from a hierarchically higher structure e.g. as one proposed in [23].[2]

Finally, if all attempts to get a hermitian representation fail this can mean that the Nambu bracket does not provide a natural environment for 3-algebra representations.

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# References

[1] J. Bagger and N. Lambert, *Comments on multiple M2-branes*, JHEP 02 (2008), 105.

[2] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, Phys. Rev. D77 (2008), 065008.

[3] J. Bagger and N. Lambert, *Modeling multiple M2's*, Phys. Rev. D75 (2007), 045020.

[4] I.A. Bandos and J.A. de Azcarraga, *BPS preons and the AdS-M-algebra*, JHEP 04 (2008), 069.

[5] R. Chatterjee, *Dynamical symmetries and Nambu mechanics*, Lett. Math. Phys. 36 (1996), 117–126.

[6] S. Cecotti and A. Sen, *Coulomb branch of the Lorentzian three algebra theory*, (Unpublished preprint arXiv:0806.1990).

[2] I thank M.M. Sheikh-Jabbari for pointing me to this possibility.

[7] R. Chatterjee and L. Takhtajan, *Aspects of classical and quantum Nambu mechanics*, Lett. Math. Phys. 37 (1996), 475–482.

[8] T. Curtright and C.K. Zachos, *Classical and quantum Nambu mechanics*, Phys. Rev. D68 (2003), 085001.

[9] J. A. de Azcarraga, J. M. Izquierdo and J. C. Perez Bueno, *On the generalizations of Poisson structures*, J. Phys. A30 (1997), L607–L616.

[10] J. A. De Azcarraga and J. C. Perez Bueno, *Higher-order simple Lie algebras*, Commun. Math. Phys. 184 (1997), 669–681.

[11] J. A. de Azcarraga, A. M. Perelomov and J. C. Perez Bueno, *The Schouten-Nijenhuis bracket, cohomology and generalized Poisson structures*, J. Phys. A29 (1996), 7993–8010.

[12] G. Dito, M. Flato, D. Sternheimer and L. Takhtajan, *Deformation quantization and Nambu mechanics*, Commun. Math. Phys. 183 (1997), 1–22.

[13] P. De Medeiros, J.M. Figueroa-O'Farrill and E. Mendez-Escobar, *Lorentzian Lie 3-algebras and their Bagger-Lambert moduli space*, JHEP 07 (2008), 111.

[14] J.P. Gauntlett and J.B. Gutowski, *Constraining maximally supersymmetric membrane actions*, (Unpublished preprint arXiv:0804.3078).

[15] J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert Theory for general Lie algebras*, JHEP 06 (2008), 075.

[16] A. Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl. Phys. B811 (2009), 66–76.

[17] A. Gustavsson, *Selfdual strings and loop space Nahm equations*, JHEP 04 (2008), 083.

[18] P.-M. Ho, R.-C. Hou and Y. Matsuo, *Lie 3-Algebra and multiple M2-branes*, JHEP 06 (2008), 020.

[19] P.-M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, *M5-brane in three-form flux and multiple M2-branes*, JHEP 08 (2008), 014.

[20] Y. Kawamura, *Cubic matrix, generalized spin algebra and uncertainty relation*, Prog. Theor. Phys. 110 (2003), 579–587.

[21] C. Krishnan and C. Maccaferri, *Membranes on Calibrations*, JHEP 07 (2008), 005.

[22] H. Lin, *Kac-Moody Extensions of 3-Algebras and M2-branes*, JHEP 07 (2008), 136.

[23] J. M. Maldacena, M.M. Sheikh-Jabbari and M. Van Raamsdonk, *Transverse fivebranes in matrix theory*, JHEP 01 (2003), 038.

[24] Y. Nambu, *Generalized Hamiltonian dynamics*, Phys. Rev. D7 (1973), 2405–2414.

[25] G. Papadopoulos, *M2-branes, 3-Lie Algebras and Plucker relations*, JHEP 05 (2008), 054.

[26] G. Papadopoulos, *On the structure of k-Lie algebras*, Class. Quant. Grav. 25 (2008), 142002.

[27] F. Passerini, *M2-brane superalgebra from Bagger-Lambert theory*, JHEP 08 (2008), 062.

[28] L. Takhtajan, *On Foundation of the generalized Nambu mechanics (second version)*, Commun. Math. Phys. 160 (1994), 295–316.



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