Factorization method and Gegenbauer polynomials in tense brane black hole

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Abstract. Using the Gegenbauer Polynomials (associated Jacobi differential) equation, we obtain the exact solution for the fermion excitations of a tense brane black hole. According to the supersymmetry approaches in quantum mechanics we obtain the first order operators which are represented by the generators of Heisenberg algebra. These generators completely relate to the $N = 2$ supersymmetry algebra.

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1 Introduction

Much has been said about black holes in large extra-dimensional scenarios recently [3] and the effect of the brane tension for black holes with large extra dimensions has largely been ignored, because of the obvious difficulty of how to embed a black hole onto a brane.
Recently Kaloper and Kiley [7] presented the following metric for a black hole on tensional 3-brane embedded in a six-dimensional space time,

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_4{}^2, \qquad g(r) = 1 - (\frac{r_H}{r})^3, \tag{1.1}$$

where the radius of the horizon is given by,

$$r_H = (\frac{\mu}{b})^{\frac{1}{3}}, \qquad \mu = \frac{M}{4\pi M_*{}^4}, \tag{1.2}$$

and M is the mass of the black hole. The parameter b is a measure of the conical deviation from a perfect sphere and has the following angle element:

$$d\Omega_4{}^2 = d\theta_3{}^2 + \sin^2\theta_3 \left[d\theta_2{}^2 + \sin^2\theta_2(d\theta_1{}^2 + b^2 \sin^2\theta_1 d\phi^2)\right], \tag{1.3}$$

where for $b = 1$ this is the line element of the unit sphere S^4 and corresponds to zero brane tension. We note here the location of the deficit angle is arbitrary such

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angle be interest to the fermion generation [4, 1]. In case of $b < 1$ we have non - vanishing brane tension, it is also measure of the deficit angle about an axis parallel to the 3-brane in the angular direction ϕ. Also it can be expressed in term of the brane tension λ as:

$$b = 1 - \frac{\lambda}{4\pi {M_*}^4}, \tag{1.4}$$

where M_* is the fundamental Planck constant of six - dimensional gravity. In the other hand, we should stress that the supersymmetry in quantum mechanics is based upon the factorization method in the framework of shape invariance. If one quantum mechanics problem obtains context supersymmetry, we must then factorize the Hamiltonian of quantum states in terms of a multiplication of the first - order differential operators as a shape invariance equations. In this approach, the Hamiltonian is decomposed once in successive multiplication of lowering and raising operators, in such a way the corresponding quantum states of successive levels, are their eigenstates. These Hamiltonian are called partner and supersymmetry of each other. As yet, according to the factorization method, many studies on the one - dimensional shape invariance potential in the framework of supersymmetric quantum mechanics have been carried out [6] . Nowadays the concept of shape invariance has extended to the ordinary differential equations and on this basis a second - order differential operator will decompose the multiplication of ladder operators [5, 9, 8].In this paper we apply the factorization method and obtain the factorized corresponding equation for the fermion excitations of a tense brane black hole. These first order equation leads us to obtain the raising and lowering relations with respect to n and m.

2 Deficit 2-sphere

Here, first we consider the case of a deficit 2-sphere, then we can generate results for the $(d-2)$-sphere. By using [3, 4, 1] and the following metric for two sphere one can obtain the corresponding equation,

$$ds^2 = d\theta^2 + b^2 \sin^2\theta d\phi^2, \tag{2.1}$$

where b is a positive real number and $b = 1$ represents a regular two sphere. The Dirac operator is given by,

$$\gamma^c \nabla_c \psi = \gamma^c e_c^\mu \left(\partial_\mu + \Gamma_\mu\right)\psi, \tag{2.2}$$

where the spin connection Γ_μ is given in terms of the zweibein e_c^μ and its inverse,

$$\Gamma_\mu = \frac{1}{8}\left[\gamma^c, \gamma^f\right] e_c^\nu \left(\partial_\mu e_{f\nu} - \Gamma^c_{\mu\nu} e_{fc}\right), \tag{2.3}$$

where $\Gamma^c_{\mu\nu}$ is the Christoffel symbol. For the above metric, the only non - vanishing $\Gamma^c_{\mu\nu}$ are,

$$\Gamma^\theta_{\phi\phi} = -b^2 \sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta, \tag{2.4}$$

Choosing the zweibein to be

$$e^c_\mu = diag(1, b\sin\theta), \qquad e^\mu_c = diag(1, \frac{1}{b\sin\theta}), \tag{2.5}$$

and the Dirac matrices

$$\gamma^c = \sigma^1; \qquad \gamma^\phi = \sigma^2, \tag{2.6}$$

the spin connection are found to be

$$\Gamma_\theta = 0 \qquad \Gamma_\phi = -\frac{i}{2} b\cos\theta\sigma^3, \tag{2.7}$$

where σ^i are the Pauli matrices. Now the Dirac operator can be written down explicitly as,

$$\left[\sigma^1\partial_\theta + \sigma^2\frac{1}{b\sin\theta}(\partial_\phi + \Gamma_\phi)\right]\Psi = \left[\sigma^1(\partial_\theta + \frac{1}{2}\cot\theta) + \sigma^2\frac{1}{b\sin\theta}\partial_\phi\right]\Psi. \tag{2.8}$$

Suppose we write the eigenvalue of this operator as $\pm iK$ and express the fermion field Ψ in two component form,

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}. \tag{2.9}$$

Then we find the following set of equations:

$$\left[\sigma^1(\partial_\theta + \frac{1}{2}\cot\theta) + \sigma^2\frac{1}{b\sin\theta}\partial_\phi\right]\Psi_\pm = \pm iK\Psi_\pm. \tag{2.10}$$

Let us consider Ψ_+, where Ψ_- can be dealt with analogously. Consider the equation for ∂_ϕ

$$\partial_\phi\chi^{(\pm)}_m = \pm im\chi^{(\pm)}_m. \tag{2.11}$$

Note that for spinors, one should get a sign change for a 2π rotation in ϕ. Therefore, the eigenvalues of m should be, $m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\cdots$.
Returning to the eigenvalue for ψ_+ we can make the following spinor separation of variables ansatz:

$$\Psi^{(\pm)}_{+nm} = \begin{pmatrix} A^{(\pm)}_n(\theta)\chi^{(\pm)}_m(\phi) \\ B^{(\pm)}_n(\theta)\chi^{(\pm)}_m(\phi) \end{pmatrix}. \tag{2.12}$$

Putting this into the eigenvalue equation, we have

$$\begin{aligned} \left[(\partial_\theta + \frac{1}{2}\cot\theta) \mp \frac{m}{b\sin\theta}\partial_\phi\right] A^{(\pm)}_n &= \pm iKB^{(\pm)}_n \\ \left[(\partial_\theta + \frac{1}{2}\cot\theta) \mp \frac{m}{b\sin\theta}\partial_\phi\right] B^{(\pm)}_n &= \pm iKA^{(\pm)}_n. \end{aligned} \tag{2.13}$$

So, the corresponding second order equation will be as,

$$A''(\theta) + \cot\theta A'(\theta) + \left[K^2 - \frac{m^2}{b^2\sin^2\theta} - \frac{1}{4}\cot^2\theta + \frac{m}{b\sin\theta}\cot\theta - \frac{1}{2}\right] A(\theta) = 0. \tag{2.14}$$

In here we take the following change of variables,

$$\cos\theta = x, \qquad A(x) = u(x) P^{(\alpha,\beta)}_{n,m}(x), \tag{2.15}$$

so, we have

$$\begin{aligned} &(1-x^2)P''^{(\lambda)}_{n,m}(x) + \left[\frac{2u'}{u}(1-x^2) - 2x\right] P'^{(\lambda)}_{n,m}(x) \\ + &\left[(1-x^2)\frac{u''}{u} - 2x\frac{u'}{u} + K^2 - \frac{m'^2}{b^2(1-x^2)} - \frac{1}{4}\frac{x^2}{(1-x^2)} + \frac{mx}{b(1-x^2)} - \frac{1}{2}\right] P^{(\lambda)}_{n,m}(x) = 0. \end{aligned} \tag{2.16}$$

3 Raising and lowering operators

Substituting the explicit forms of the raising and lowering operators

$$\begin{aligned} A^+_n(x) &= (1-x^2)\frac{d}{dx} - (2\lambda + n)x \\ A^-_n(x) &= -(1-x^2)\frac{d}{dx} - nx, \end{aligned} \tag{3.1}$$

for a given m, we can factorize the associated differential equation (2.16) with considering the equation (3.1) with respect to the parameter n:

$$\begin{aligned} A^+_n(x)A^-_n(x)P^{(\lambda)}_{n,m}(x) &= (n-m)(2\lambda+n+m)P^{(\lambda)}_{n,m}(x) \\ A^-_n(x)A^+_n(x)P^{(\lambda)}_{n-1,m}(x) &= (n-m)(2\lambda+n+m)P^{(\lambda)}_{n-1,m}(x). \end{aligned} \tag{3.2}$$

Also we can write the above equation in form of raising and lowering equations with respect to the parameter n as below

$$\begin{aligned} A^+_n(x)P^{(\lambda)}_{n-1,m}(x) &= \sqrt{(n-m)(2\lambda+n+m)}P^{(\lambda)}_{n,m}(x) \\ A^-_n(x)P^{(\lambda)}_{n,m}(x) &= \sqrt{(n-m)(2\lambda+n+m)}\,P^{(\lambda)}_{n-1,m}(x). \end{aligned} \tag{3.3}$$

For a given n, the differential equation (2.16) can be also factorized with respect to m as the following shape invariance equations

$$\begin{aligned} A^+_m(x)A^-_m(x)P^{(\lambda)}_{n,m}(x) &= (n-m+1)(2\lambda+n+m)P^{(\lambda)}_{n,m}(x) \\ A^-_m(x)A^+_m(x)P^{(\lambda)}_{n,m-1}(x) &= (n-m+1)(2\lambda+n+m)P^{(\lambda)}_{n,m-1}(x), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} A^+_m(x) &= \sqrt{1-x^2}\frac{d}{dx} + \frac{(m-1)x}{\sqrt{1-x^2}} \\ A^-_m(x) &= -\sqrt{1-x^2}\frac{d}{dx} + \frac{(2\lambda+m)x}{\sqrt{1-x^2}}. \end{aligned} \tag{3.5}$$

In contrast to the previous case, the raising and lowering operators i.e. $A^+_m(x)$ and $A^-_m(x)$ are Hermitian conjugate of each other respect to the inner product. The equations (3.3) can be written as the raising and lowering relations of the associated Gegenbauer functions:

$$\begin{aligned} A^+_m(x)P^{(\lambda)}_{n,m-1}(x) &= (n-m+1)(2\lambda+n+m)P^{(\lambda)}_{n,m}(x) \\ A^-_m(x)P^{(\lambda)}_{n,m}(x) &= (n-m+1)(2\lambda+n+m)P^{(\lambda)}_{n,m-1}(x) \end{aligned} \tag{3.6}$$

4 Conclusions

In this paper we used the factorization method and obtained the factorized corresponding equation for the fermion excitations of a tense brane black hole . These first order equation leads us to obtain the raising and lowering relations with respect to n and m. Also these operators give us the generators of algebra and some representation in Physics.

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