A fixed point theorem for w-distance

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Abstract. In this paper, a fixed point theorem for a (Φ, Ψ, p)-contractive map on a complete metric space is proved. In other words, let p be a w-distance on a complete metric space (X, d) and S a (φ, ψ, p)-contractive map on X [i.e. for each $x, y \in X, \varphi p(Sx, Sy) \leq \psi \varphi p(x, y)$], then S has a unique fixed point in X. Moreover, $lim_n S^n x$ is a fixed point of S for each $x \in X$.

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1 Introduction

Branciari [1] established a fixed point result for an integral-type inequality, which is a generalization of Banach contraction principle. Vijayaraju et al. [4] obtained a general principle, which made it possible to prove many fixed point theorems for pairs of integral type maps. Kada et al. [2] defined the concept of w-distance in a metric space and studied some fixed point theorems.
Now, we prove a fixed point theorem which is a new version of the main theorem in [1], by considering the concept of the w-distance, and as a result of it, we can have the main theorem of [1]. In order to do this, we recall some definitions and lemmas form [1], [2] and [3].

Definition 1.1. Let X be a metric space with metric d. A function $p : X \times X \longrightarrow [0, \infty)$ is called a w-distance on X if
(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(2) p is lower semi-continuous in its second variable i.e. if $x \in X$ and $y_n \to y$ in X then $p(x, y) \leq liminf_n p(x, y_n)$;
(3) For each $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) \leq \epsilon$.

Example 1.2. *If $X = \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$. For each $x, y \in X$, $d(x, y) = x + y$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$ is a metric on X and (X, d) is a complete metric space. Moreover, by defining $p(x, y) = y$, p is a w-distance on (X, d).*

Suppose
$$\Phi = \{\varphi | \varphi : [0, \infty) \to [0, \infty)\}$$

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where φ is non-decreasing, continuous and $\varphi(\varepsilon) > 0$ for each $\varepsilon > 0$. Moreover, let

$$\Psi = \{\psi|\psi : [0, \infty) \to [0, \infty)\}$$

where ψ is non-decreasing, right continuous and $\psi(t) < t$ for all $t > 0$.

Example 1.3. *Let $\{a_n\}_{n=1}^{\infty} and \{c_n\}_{n=0}^{\infty}$ are two non-negative sequences such that $\{a_n\}$ strictly decreasing, convergence to zero, and for each $n \in \mathbb{N}, c_{n-1}a_n > a_{n+1}$ where $0 < c_{n-1} < 1$ define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(0) = 0, \psi(t) = c_n t$, if $a_{n+1} \leq t < a_n, \psi(t) = c_0 t$ if $t \geq a_1$, then ψ is in Ψ.*

Now, we prove the following lemma.

Lemma 1.4. *If $\psi \in \Psi$ then $lim_n \psi^n(t) = 0$ for each $t > 0$ and if $\varphi \in \Phi, \{a_n\} \subseteq [0, \infty)$ and $lim_n \varphi(a_n) = 0$ then $lim_n a_n = 0$.*

Proof. For each $t > 0, \{\psi^n(t)\}$ is decreasing non-negative sequence thus there exists $\alpha \geq 0$ such that $\alpha = lim_n \psi^n(t)$ or $\psi^n(t) \to \alpha^+$ as n$\to \infty$ but ψ is right continuous in α thus $\psi^{n+1}(t) \to \psi(\alpha)$ as $n \to \infty$ thus $\psi(\alpha) = \alpha$ and therefore $\alpha = 0$. If there exists $\varepsilon > 0$ and $\{n_k\}_{k=1}^{\infty}$ such that

$$a_{n_k} \geq \varepsilon > 0$$

then

$$lim_k sup \varphi(a_{n_k}) \geq \varphi(\varepsilon) > 0$$

thus $lim_n \varphi(a_n) \neq 0$. □

The following two lemmas are used in the next section.

Lemma 1.5. *[2] Let (X, d) be a metric space and p be a w-distance on X. If $\{x_n\}$ is a sequence in X such that $lim_n p(x_n, x) = lim_n p(x_n, y) = 0$ then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$ then $x = y$.*

If $p(a, b) = p(b, a) = 0$ and $p(a, a) \leq p(a, b) + p(b, a) = 0$, then $p(a, a) = 0$ and by Lemma 1.5 $a = b$.

Lemma 1.6. *[2] Let p be a w-distance on metric space (X, d) and $\{x_n\}$ be a sequence in X such that for each $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $p(x_n, x_m) < \varepsilon$ (or $\lim_{m,n} p(x_n, x_m) = 0$) then $\{x_n\}$ is a Cauchy sequence.*

2 Main Result

In this section, we state the main theorem as follows:

Theorem 2.1. *Let p be a w-distance on a complete metric space $(X, d), \varphi \in \Phi$ and $\psi \in \Psi$. Suppose S is a (φ, ψ, p)- contractive map on X [i.e. for each $x, y \in X, \varphi p(Sx, Sy) \leq \psi \varphi p(x, y)$] then S has a unique fixed point in X. Moreover, $lim_n S^n x$ is a fixed point of S for each $x \in X$.*

Proof. Fix $x \in X$. Set $x_{n+1} = Sx_n$ with $x_0 = x$. Then

$$\begin{aligned}\varphi p(x_n, x_{n+1}) \leq &\ \psi\varphi p(x_{n-1}, x_n)\\ \leq &\ \psi^2\varphi p(x_{n-2}, x_{n-1})\\ \leq &\ \cdots\\ \leq &\ \psi^n(\varphi p(x_0, x_1)).\end{aligned}$$

Thus $\lim_n \varphi p(x_n, x_{n+1}) = 0$ and Lemma 1.4 implies

$$\lim_n p(x_n, x_{n+1}) = 0 \tag{2.1}$$

and similarly

$$\lim_n p(x_{n+1}, x_n) = 0 \tag{2.2}$$

step1: $\lim_{m,n} p(x_n, x_m) = 0$.
proof of step1: Suppose there exists $\varepsilon > 0$ and $\{m_k\}_{k=1}^{\infty}, \{n_k\}_{k=1}^{\infty}$ such that

$$p(x_{n_k}, x_{m_k}) \geq \varepsilon \tag{2.3}$$

where $m_k > n_k$. By (2.1) there exists $k_0 \in \mathbb{N}$ such that $n_k > k_0$ implies

$$p(x_{n_k}, x_{n_{k+1}}) < \varepsilon. \tag{2.4}$$

If $n_k > k_0$ by (2.3) and (2.4), $m_k \neq n_{k+1}$. We can assume that m_k is a minimal index such that $p(x_{n_k}, x_{m_k}) \geq \varepsilon$ but $p(x_{n_k}, x_h) < \varepsilon, h \in \{n_{k+1}, \cdots, m_k - 1\}$.
We have

$$\begin{aligned}\varepsilon \leq &\ p(x_{n_k}, x_{m_k})\\ \leq &\ p(x_{n_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{m_k})\\ < &\ \varepsilon + p(x_{m_k-1}, x_{m_k}) \to \varepsilon^+\end{aligned}$$

as $k \to \infty$ this implies $\lim_k p(x_{n_k}, x_{m_k}) = \varepsilon^+$. If $\eta := \limsup_k p(x_{n_k+1}, x_{m_k+1}) \geq \varepsilon$ then there exists $\{k_r\}_{r=1}^{\infty}$ such that

$$p(x_{n_{k_r}+1}, x_{m_{k_r}+1}) \to \eta \geq \varepsilon$$

as $r \to \infty$. Since φ is continuous and non- decreasing

$$\begin{aligned}\varphi(\varepsilon) \leq \varphi(\eta) = &\ \lim_r \varphi p(x_{n_{k_r}+1}, x_{m_{k_r}+1})\\ \leq &\ \lim_r \psi\varphi p(x_{n_{k_r}}, x_{m_{k_r}})\\ = &\ \psi\varphi(\varepsilon).\end{aligned}$$

Note that

$$\varphi p(x_{n_{k_r}}, x_{m_{k_r}}) \to \varphi(\varepsilon)^+$$

and ψ is right continuous. Thus $\varphi(\varepsilon) = 0$. This is a contradiction and $\limsup_k p(x_{n_k+1}, x_{m_k+1}) < \varepsilon$, so we have

$$\begin{aligned}\varepsilon \leq &\ p(x_{n_k}, x_{m_k})\\ \leq &\ p(x_{n_k}, x_{n_k+1}) + p(x_{n_k+1}, x_{m_k+1}) + p(x_{m_k+1}, x_{m_k}).\end{aligned}$$

Then (2.1) and (2.2) implies that

$$\begin{aligned}\varepsilon \leq &\ \lim_k p(x_{n_k}, x_{n_k+1}) + \limsup_k p(x_{n_k+1}, x_{m_k+1}) + \lim_k p(x_{m_k+1}, x_{m_k})\\ = &\ \limsup_k p(x_{n_k+1}, x_{m_k+1}) < \varepsilon\end{aligned}$$

which is a contradiction. Now, we proved

$$\lim_{m,n} p(x_n, x_m) = 0. \tag{2.5}$$

By Lemma 1.6, $\{x_n\}$ is a Cauchy sequence and since X is complete, there exists $u \in X$ such that $x_n \to u$ in X.

step2: u is a fixed point of S.

proof of step2: By (2.5) for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon$ implies $p(x_{N_\varepsilon}, x_n) < \varepsilon$ but $x_n \to u$ and $p(x, .)$ is lower semi continuous thus

$$p(x_{N_\varepsilon}, u) \leq \liminf_n p(x_{N_\varepsilon}, x_n) \leq \varepsilon.$$

Therefore $p(x_{N_\varepsilon}, u) \leq \varepsilon$. Set $\varepsilon = 1/k, N_\varepsilon = n_k$ and we have

$$\lim_k p(x_{n_k}, u) = 0. \tag{2.6}$$

On the other hand,

$$\varphi p(x_{n_k+1}, Su) \leq \psi\varphi p(x_{n_k}, u) \to 0$$

(as $k \to \infty$) and thus $\lim_k p(x_{n_k+1}, Su) = 0$, but

$$p(x_{n_k}, Su) \leq p(x_{n_k}, x_{n_k+1}) + p(x_{n_k+1}, Su)$$

thus

$$\lim_k p(x_{n_k}, Su) = 0. \tag{2.7}$$

Now (2.6), (2.7) and Lemma 1.5 implies that $Su = u$.

step3: The fixed point of S is unique.

proof of step3: Suppose u_1 and u_2 are two arbitrary fixed points of S. We have

$$\varphi p(u_1, u_2) = \varphi p(Su_1, Su_2) \leq \psi\varphi p(u_1, u_2)$$

Thus $\varphi p(u_1, u_2) = 0$ and $p(u_1, u_2) = 0$. Similarly, $p(u_2, u_1) = 0$ and then $u_1 = u_2$. $\square$

Remark 2.2. *(1) In the above theorem, let $p = d, \varphi(t) = t, \psi(t) = ct(c \in [0, 1[)$. Then Theorem 2.1 is the classical Banach fixed point theorem.*
(2) Suppose $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue- integrable mapping which is summable and $\int_0^\varepsilon \theta(\eta)d\eta > 0$ for each $\varepsilon > 0$. Set $\varphi(t) = \int_0^t \theta(\eta)d\eta$ and $\psi(t) = ct$, where $c \in [0, 1[$. Then $\varphi \in \Phi$ and the main theorem of [1] is obtained.

Remark 2.3. *If p and d are in example 1.2, $\varphi \in \Phi$ and $\psi \in \Psi$ then for each $x, y \in X(y \neq 0), p(x, y) = y = d(0, y)$ thus each (φ, ψ, d)-contractive map is (φ, ψ, p)-contractive map. But, the converse is not valid.*

Example 2.4. *Let (X, d) and p be in Example 1.2 and $S : X \to X$ be a map as $S\frac{1}{n} = \frac{1}{n+1}$, $S0 = 0$. Suppose $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and strictly non-decreasing map. Let ψ be the map in Example 1.3 as $a_n = \varphi(\frac{1}{n})$. Moreover, we assume that*

$$\varphi(\tfrac{1}{n+1})\varphi(\tfrac{1}{n} + \tfrac{1}{n+1}) < \varphi(\tfrac{1}{n})\varphi(\tfrac{1}{n+1} + \tfrac{1}{n+2}),$$

(for example $\varphi(t) = t$), then S is (φ, ψ, p)-contractive, since

$$\varphi p(S\frac{1}{m}, S\frac{1}{n}) = \varphi(\frac{1}{n+1}) < c_{n-1}\varphi(\frac{1}{n}) = \psi\varphi p(\frac{1}{m}, \frac{1}{n}),$$

where

$$\frac{\varphi(\frac{1}{n+1})}{\varphi(\frac{1}{n})} < c_{n-1} < \frac{\varphi(\frac{1}{n+1} + \frac{1}{n+2})}{\varphi(\frac{1}{n} + \frac{1}{n+1})}$$

but S is not (φ, ψ, d)-contractive, since

$$\begin{aligned} \varphi d(S\tfrac{1}{n}, S\tfrac{1}{n+1}) = &\ \varphi(\tfrac{1}{n+1} + \tfrac{1}{n+2}) > c_{n-1}\varphi(\tfrac{1}{n} + \tfrac{1}{n+1}) \\ = &\ \psi\varphi(\tfrac{1}{n} + \tfrac{1}{n+1}) = \psi\varphi d(\tfrac{1}{n}, \tfrac{1}{n+1}). \end{aligned}$$

3 Conclusion

In this paper, a fixed point theorem for a (ϕ, ψ, p)- contractive map was proved. As a result, the classical Banach fixed point theorem was obtained. Moreover, the main theorem of [1] which proves a fixed point theorem for a general contractive condition of integral type will be obtained from the main theorem of this paper. Finally, an example was given to prove the validity of the theorem.

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