

# Galilean geometry of motions

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**Abstract.** In this paper we show that Galilean group is a matrix Lie group and find its structure. Then provide the invariants of special Galilean geometry of motions, by Olver's method of moving coframes, we also find its corresponding  $\{e\}$ -structure.

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**Key words:** Galilean motion, Cartan equivalence method, moving coframe.

## 1 Introduction:

In 1998-1999 *M. Fels* and *P.J. Olver* in [1] and [2], gave a practical method for finding the invariants of the equivalence problem via moving frames. The interest therefore is to find an applicable method in lieu of predecessors, *G. Darboux*, *E. Cartan* and *R. Gardner*. By using this method, we find all differential invariants of special Galilean  $\text{SGal}(3)$ , in the theorems [16], [17], and [18], then the necessary condition will be explained. Let us itemize the basic steps of moving coframes method.

The basic steps are:

- (i) Determine the moving frame of order zero, by choosing a base point and solving for Galilean group action.
- (ii) Determine the invariant forms in this case (the finite dimensional), they are the Maurer-Cartan forms, which computed by direct use of the transformation group formulae and not the matrix method.
- (iii) Use the invariant lift to pull-back the invariant forms, leading to the moving coframe of order zero.
- (iv) Determine the lifted invariants by finding the linear dependencies among the restricted to horizontal components of the moving coframe forms.
- (v) Normalize any group-dependent invariants to convenient constant values by solving for some of the unspecified parameters.
- (vi) Successively eliminate parameters by substituting the normalization formulae into the moving coframe and recomputing dependencies.

- (vii) After the parameters have all been normalized, the differential invariants will appear through any remaining dependencies among the final moving coframe elements. The Invariant differential operators are found as the dual differential operators to a basis for the invariant coframe forms.

From now on, we define some basic prerequisites from Galilean group. Explanatory details are found in [3, 4, 9].

**Definition 1.** The Galilean group is defined as

$$(1.1) \quad \text{Gal}(3) := \left\{ \left[ \begin{array}{ccc} 1 & 0 & s \\ v & R & y \\ 0 & 0 & 1 \end{array} \right] \middle| s \in \mathbb{R}, y, v \in \mathbb{R}^3, R \in O(3) \right\}$$

with a natural closed Lie subgroup structure of  $GL(5; \mathbb{R})$  as

$$(1.2) \quad \begin{aligned} & \left[ \begin{array}{ccc} 1 & 0 & s_1 \\ v_1 & R_1 & y_1 \\ 0 & 0 & 1 \end{array} \right] \cdot \left[ \begin{array}{ccc} 1 & 0 & s_2 \\ v_2 & R_2 & y_2 \\ 0 & 0 & 1 \end{array} \right] = \\ & = \left[ \begin{array}{ccc} 1 & 0 & s_1 + s_2 \\ v_1 + R_1 v_2 & R_1 R_2 & y_1 + s_2 v_1 + R_1 y_2 \\ 0 & 0 & 1 \end{array} \right] \\ & \left[ \begin{array}{ccc} 1 & 0 & s \\ v & R & y \\ 0 & 0 & 1 \end{array} \right]^{-1} = \left[ \begin{array}{ccc} 1 & 0 & -s \\ -R^{-1}v & R^{-1} & R^{-1}(sv - y) \\ 0 & 0 & 1 \end{array} \right]. \end{aligned}$$

This is a 10-dimensional Lie group. The *special Galilean group* is defined as connected component of  $e$  in  $\text{Gal}(3)$  and denoted by  $\text{SGal}(3)$ .

**Definition 2.** Let us identify the  $\mathbb{R}^4$  by

$$(1.3) \quad \mathbb{R}^4 := \left\{ \left[ \begin{array}{c} t \\ x \\ 1 \end{array} \right] \middle| t \in \mathbb{R}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

with the natural 4-manifold structure. Then, we can define naturally the smooth action of  $\text{Gal}(3)$  on  $\mathbb{R}^4$  as

$$(1.4) \quad \left[ \begin{array}{ccc} 1 & 0 & s \\ v & R & y \\ 0 & 0 & 1 \end{array} \right] \bullet \left[ \begin{array}{c} t \\ x \\ 1 \end{array} \right] := \left[ \begin{array}{c} t + s \\ Rx + tv + y \\ 1 \end{array} \right]$$

By elementary algebraic computations, we find the structure of special Galilean group,

**Theorem 1.** Let

$$G_1 := \left\{ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ v & R & 0 \\ 0 & 0 & 1 \end{array} \right] \middle| v \in \mathbb{R}^3, R \in O(3) \right\} \leq GL(4; \mathbb{R})$$

be the group of uniformly special Galilean motions,

$$(1.5) \quad G_2 := \left\{ \left[ \begin{array}{ccc} 1 & 0 & s \\ 0 & I_3 & y \\ 0 & 0 & 1 \end{array} \right] \middle| s \in \mathbb{R}, y \in \mathbb{R}^3 \right\} \cong (\mathbb{R}^4, +)$$

be the group of shifts of origin,

$$(1.6) \quad G_3 := \left\{ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 1 \end{array} \right] \middle| R \in \text{SO}(3) \right\} \cong \text{SO}(3)$$

be the group of rotations of reference frame, and

$$(1.7) \quad G_4 := \left\{ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ v & I_3 & 0 \\ 0 & 0 & 1 \end{array} \right] \middle| v \in \mathbb{R}^3 \right\} \cong (\mathbb{R}^3, +)$$

be the group of uniformly frame motions. Then,  $G_2 \trianglelefteq \text{SGal}(3)$ ,  $\text{SGal}(3) \cong G_1 \times G_2$ ,  $G_4 \trianglelefteq G_1$ ,  $G_1 \cong G_3 \times G_4$ , and  $\text{SGal}(3) \cong (\text{SO}(3) \times \mathbb{R}^3) \times \mathbb{R}^4$ .

In the following theorem, we explain the algebraic structure of the infinitesimal group action  $\widehat{\mathfrak{Gal}}(3)$  induced by the action  $\text{SGal}(3)$  on  $\mathbb{R}^4$ ,

**Theorem 2.** *The Lie algebra of infinitesimal group action  $\widehat{\mathfrak{Gal}}(3) = \text{Span}_{\mathbb{R}}\{\widehat{X}_1, \dots, \widehat{X}_{10}\}$  induced by the action  $\text{SGal}(3)$  on  $\mathbb{R}^4$ , has infinitesimal generators:*

$$(1.8) \quad \begin{array}{lll} \widehat{X}_2 := \partial_x, & \widehat{X}_5 := t \partial_x, & \widehat{X}_8 := y \partial_x - x \partial_y, \\ \widehat{X}_1 := \partial_t, & \widehat{X}_3 := \partial_y, & \widehat{X}_6 := t \partial_y, & \widehat{X}_9 := x \partial_z - z \partial_x, \\ & \widehat{X}_4 := \partial_z, & \widehat{X}_7 := t \partial_z, & \widehat{X}_{10} := z \partial_y - y \partial_z, \end{array}$$

with the following structure:

	$\widehat{X}_1$	$\widehat{X}_2$	$\widehat{X}_3$	$\widehat{X}_4$	$\widehat{X}_5$	$\widehat{X}_6$	$\widehat{X}_7$	$\widehat{X}_8$	$\widehat{X}_9$	$\widehat{X}_{10}$
$\widehat{X}_1$	0	0	0	0	$\widehat{X}_2$	$\widehat{X}_3$	$\widehat{X}_4$	0	0	0
$\widehat{X}_2$	0	0	0	0	0	0	0	$-\widehat{X}_3$	$\widehat{X}_4$	0
$\widehat{X}_3$	0	0	0	0	0	0	0	$\widehat{X}_2$	0	$-\widehat{X}_4$
$\widehat{X}_4$	0	0	0	0	0	0	0	0	$-\widehat{X}_2$	$\widehat{X}_3$
$\widehat{X}_5$	$-\widehat{X}_2$	0	0	0	0	0	0	$-\widehat{X}_6$	$\widehat{X}_7$	0
$\widehat{X}_6$	$-\widehat{X}_3$	0	0	0	0	0	0	$\widehat{X}_5$	0	$-\widehat{X}_7$
$\widehat{X}_7$	$-\widehat{X}_4$	0	0	0	0	0	0	0	$-\widehat{X}_5$	$\widehat{X}_6$
$\widehat{X}_8$	0	$\widehat{X}_3$	$-\widehat{X}_2$	0	$\widehat{X}_6$	$-\widehat{X}_5$	0	0	$-\widehat{X}_{10}$	$\widehat{X}_9$
$\widehat{X}_9$	0	$-\widehat{X}_4$	0	$\widehat{X}_2$	$-\widehat{X}_7$	0	$\widehat{X}_5$	$\widehat{X}_{10}$	0	$-\widehat{X}_8$
$\widehat{X}_{10}$	0	0	$\widehat{X}_4$	$-\widehat{X}_3$	0	$\widehat{X}_7$	$-\widehat{X}_6$	$-\widehat{X}_9$	$\widehat{X}_8$	0

## 2 Computation of Maurer-Cartan forms

**Definition 3.** By multiplying 3 rotations

$$\begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{bmatrix}$$

respectively about  $z$ ,  $y$  and  $x$ -axis, we define  $R$  as

$$\begin{bmatrix} \cos \theta_2 \cos \theta_3 & \cos \theta_1 \sin \theta_3 - \sin \theta_1 \sin \theta_2 \cos \theta_3 & \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \\ -\cos \theta_2 \sin \theta_3 & \cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3 & \sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3 \\ -\sin \theta_2 & -\sin \theta_1 \cos \theta_2 & \cos \theta_1 \cos \theta_2 \end{bmatrix}.$$

which is an arbitrary element of  $SO(3)$ .

In order to determine the Maurer-Cartan forms, we would rather use the direct method, more details found in page 10 of [1].

**Theorem 3.** *The independent Maurrer-Cartan 1-forms of  $\text{SGal}(3)$  are*

$$\begin{aligned}
\mu_1 &:= ds, \\
\mu_2 &:= \sin \theta_2 d\theta_1 - d\theta_3, \\
\mu_3 &:= \cos \theta_2 \cos \theta_3 d\theta_1 - \sin \theta_3 d\theta_2, \\
\mu_4 &:= \cos \theta_2 \sin \theta_3 d\theta_1 + \cos \theta_3 d\theta_2, \\
\mu_5 &:= \cos \theta_1 \cos \theta_2 dv_1 - \sin \theta_1 \cos \theta_2 dv_2 - \sin \theta_2 dv_3, \\
\mu_6 &:= \left( \cos \theta_1 \cos \theta_2 v_1 - \sin \theta_1 \cos \theta_2 v_2 - \sin \theta_2 v_3 \right) ds \\
&\quad - \cos \theta_1 \cos \theta_2 dy_1 + \sin \theta_1 \cos \theta_2 dy_2, \\
\mu_7 &:= -\left( \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \right) dv_1 \\
&\quad + \left( \sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3 \right) dv_2 - \cos \theta_2 \cos \theta_3 dv_3, \\
\mu_8 &:= \left( \cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3 \right) dv_1 \\
&\quad - \left( \sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3 \right) dv_2 + \cos \theta_2 \sin \theta_3 dv_3, \\
\mu_9 &:= \left( \left( \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \right) v_1 + \left( \cos \theta_1 \sin \theta_3 \right. \right. \\
&\quad \left. \left. - \sin \theta_1 \sin \theta_2 \cos \theta_3 \right) v_2 + \cos \theta_2 \cos \theta_3 v_3 \right) ds \\
&\quad - \left( \cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \right) dy_1 \\
&\quad + \left( -\cos \theta_1 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3 \right) dy_2 - \cos \theta_2 \cos \theta_3 dy_3, \\
\mu_{10} &:= \left( \left( \sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3 \right) v_1 + \left( \cos \theta_1 \cos \theta_3 \right. \right. \\
&\quad \left. \left. - \sin \theta_1 \sin \theta_2 \sin \theta_3 \right) v_2 + \cos \theta_2 \sin \theta_3 v_3 \right) ds \\
&\quad - \left( \cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \right) dy_1 \\
&\quad + \left( \sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3 \right) dy_2 - \cos \theta_2 \cos \theta_3 dy_3.
\end{aligned}$$

*Proof.* Given  $g \in \text{SGal}(3)$ , and  $z \in \mathbb{R}^4$  we explicitly write the group transformation  $\bar{z} = g \cdot z$  in coordinate form:

$$\begin{aligned}
\bar{z}_1 &= H^1(z, g) = t + s \\
\bar{z}_2 &= H^2(z, g) \\
&= v_1 t + (\cos \theta_2 \cos \theta_1) x_1 + (\cos \theta_3 \sin \theta_1 - \sin \theta_3 \sin \theta_2 \cos \theta_1) x_2 \\
&\quad + (\sin \theta_3 \sin \theta_1 + \cos \theta_3 \sin \theta_2 \cos \theta_1) x_3 + y_1 \\
\bar{z}_3 &= H^3(z, g) \\
&= v_2 t - (\cos \theta_2 \sin \theta_1) x_1 + (\cos \theta_3 \cos \theta_1 + \sin \theta_3 \sin \theta_2 \sin \theta_1) x_2 \\
&\quad + (\sin \theta_3 \cos \theta_1 - \cos \theta_3 \sin \theta_2 \sin \theta_1) x_3 + y_2 \\
\bar{z}_4 &= H^4(z, g) \\
&= v_3 t - \sin \theta_2 x_1 - (\sin \theta_3 \cos \theta_2) x_2 + (\cos \theta_3 \cos \theta_2) x_3 + y_3.
\end{aligned}$$

We then compute the differentials of the group transformations:

$$d\bar{z}_i = \sum_{k=1}^4 \frac{\partial H^i}{\partial z_k} dz_k + \sum_{j=1}^{10} \frac{\partial H^i}{\partial g^j} dg^j, \quad i = 1, \dots, 4,$$

or more compactly

$$(2.1) \quad d\bar{z} = H_z dz + H_g dg.$$

Then, set  $d\bar{z} = 0$  in (2.1), and solve the resulting system of linear equations for the differentials  $dz_k$ . This leads to the formulae

$$-dz = F dg = (H_z^{-1} \cdot H_g) dg,$$

or, in full detail,

$$(2.2) \quad -dz_k = \sum_{j=1}^{10} F_j^k(z, g) dg^j, \quad i = 1, \dots, 4.$$

Then, for each  $k$  and each fixed  $z_0 \in \mathbb{R}^4$ , the one-form

$$\mu_0 = \sum_{j=1}^{10} F_j^k(z_0, g) dg^j$$

is a left-invariant Maurer-Cartan form on the group  $\text{SGal}(3)$ . Alternatively, if one expands the right hand side of (2.2) in power series in  $z$ ,

$$\sum_{j=1}^{10} F_j^k(z, g) dg^j = \sum_{i=0}^{\infty} z_i \mu_i,$$

then each coefficient  $\mu_i$  also forms a left-invariant Maurer-Cartan form on  $\text{SGal}(3)$ .  $\square$

### 3 Zero order moving coframes

Throughout this paper, we remind you that  $\text{SGal}(3)$  is a 10-dimensional Lie group and  $M$  is a 4-dimensional manifold.

**Definition 4.** A smooth map  $\rho^{(0)} : M \rightarrow \text{SGal}(3)$  is called a compatible lift with base point  $z_0$  if it satisfies

$$\rho^{(0)}(z).z_0 = z, \quad z \in M.$$

Now, let  $\rho^{(0)} : M \rightarrow \text{SGal}(3)$  be a compatible lift with base point  $z_0 = [0, 0, 1]^T \in \mathbb{R}^4$ , then we have,  $s = t$  and  $y = x$ . Thus,

**Theorem 4.** *The most general zero order compatible lift has the form*

$$\rho^{(0)}(t, x; v, \theta) = \begin{bmatrix} 1 & 0 & t \\ v & R & x \\ 0 & 0 & 1 \end{bmatrix}.$$

The next step is to characterize the group transformations by a collection of differential forms. In the finite-dimensional situation that we are currently considering, these will be obtained by pulling back the left-invariant Maurer-Cartan forms  $\mu$  on  $\text{SGal}(3)$  to the zero order moving frame bundle  $\mathcal{B}_0$  using the compatible lift.

The resulting one-forms  $\zeta^{(0)} = \rho^{(0)*}\mu$  will provide an invariant coframe on  $\mathcal{B}_0$ , which we name the *moving coframe of the zero order*. The moving coframe forms  $\zeta^{(0)}$  clearly satisfy the same Maurer-Cartan structure equations. Thus

**Theorem 5.** *The zero order moving coframe is*

$$\begin{aligned}
\zeta_1^{(0)} &:= dt, \\
\zeta_2^{(0)} &:= \sin \theta_2 d\theta_1 - d\theta_3, \\
\zeta_3^{(0)} &:= \cos \theta_2 \cos \theta_3 d\theta_1 - \sin \theta_3 d\theta_2, \\
\zeta_4^{(0)} &:= \cos \theta_2 \sin \theta_3 d\theta_1 + \cos \theta_3 d\theta_2, \\
\zeta_5^{(0)} &:= \cos \theta_1 \cos \theta_2 dv_1 - \sin \theta_1 \cos \theta_2 dv_2 - \sin \theta_2 dv_3, \\
\zeta_6^{(0)} &:= \left( \cos \theta_1 \cos \theta_2 v_1 - \sin \theta_1 \cos \theta_2 v_2 - \sin \theta_2 v_3 \right) dt \\
&\quad - \cos \theta_1 \cos \theta_2 dx_1 + \sin \theta_1 \cos \theta_2 dx_2, \\
\zeta_7^{(0)} &:= -\left( \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \right) dv_1 \\
&\quad + \left( \sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3 \right) dv_2 - \cos \theta_2 \cos \theta_3 dv_3 \\
\zeta_8^{(0)} &:= \left( \cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3 \right) dv_1, \\
&\quad - \left( \sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3 \right) dv_2 + \cos \theta_2 \sin \theta_3 dv_3, \\
\zeta_9^{(0)} &:= \left( \left( \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \right) v_1 + \left( \cos \theta_1 \sin \theta_3 \right. \right. \\
&\quad \left. \left. - \sin \theta_1 \sin \theta_2 \cos \theta_3 \right) v_2 + \cos \theta_2 \cos \theta_3 v_3 \right) dt \\
&\quad - \left( \cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \right) dx_1 \\
&\quad + \left( -\cos \theta_1 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3 \right) dx_2 - \cos \theta_2 \cos \theta_3 dx_3, \\
\zeta_{10}^{(0)} &:= \left( \left( \sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3 \right) v_1 + \left( \cos \theta_1 \cos \theta_3 \right. \right. \\
&\quad \left. \left. - \sin \theta_1 \sin \theta_2 \sin \theta_3 \right) v_2 + \cos \theta_2 \sin \theta_3 v_3 \right) dt \\
&\quad - \left( \cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \right) dx_1 \\
&\quad + \left( \sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3 \right) dx_2 - \cos \theta_2 \cos \theta_3 dx_3,
\end{aligned}$$

which forms a basis for the space of one-forms on  $\mathcal{B}_0 := \mathbb{R}^4 \times G_1$  □.

## 4 First order moving coframes

**Definition 5.** A motion is a curve coincides with the graph of a function  $x = x(t) : \mathbb{R} \rightarrow \mathbb{R}^3$ .

We restrict the moving coframe forms to the motion (curve), which amounts to replacing the differential  $dx$  by its "horizontal" component  $x_t dt$ . If we interpret the

derivative  $x_t$  as a coordinate on the first jet space  $J^1 = J^1(\mathbb{R}^1; \mathbb{R}^3) \cong \mathbb{R}^7$  of curves in  $\mathbb{R}^4$ , then the restriction of a differential form to the curve can be reinterpreted as the natural projection of the one-form  $dx$  on  $J^1$  to its horizontal component, using the canonical decomposition of differential forms on the jet space into horizontal and contact components. Indeed, the vertical component of the form  $dx$  is the contact form  $dx - x_t dt$ , which vanishes on all prolonged sections of the first jet bundle  $J^1(\mathbb{R}^1; \mathbb{R}^3)$ . Therefore,

**Theorem 6.** *The restricted (or horizontal) moving coframe forms are defined on 7-dimensional manifold  $\{J^1x := (t, x(t), x_t(t))\} \times G_1 \subset J^1\mathcal{B}_0$  and explicitly given by  $\eta_i^{(0)} = \zeta_i^{(0)}$ , for  $i = 1, 2, 3, 4, 5, 7, 8$ , and their linear dependencies are  $\eta_6^{(0)} = j_1 \eta_1^{(0)}$ ,  $\eta_9^{(0)} = j_2 \eta_1^{(0)}$  and  $\eta_{10}^{(0)} = j_3 \eta_1^{(0)}$ , where*

$$\begin{aligned} J_1 &= -\cos \theta_1 \cos \theta_2 v_1 + \sin \theta_1 \cos \theta_2 v_2 + \sin \theta_2 v_3 \\ &\quad + \cos \theta_1 \cos \theta_2 x'_1 + \sin \theta_1 \cos \theta_2 x'_2 - \sin \theta_2 x'_3, \\ J_2 &= (\cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3) v_1 \\ &\quad - (\cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3) v_2 + \cos \theta_2 \sin \theta_3 v_3 \\ &\quad + (\sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3) x'_1 \\ &\quad + (\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3) x'_2 - \cos \theta_2 \sin \theta_3 x'_3, \\ J_3 &= -(\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) v_1 \\ &\quad + (\sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3) v_2 - \cos \theta_2 \cos \theta_3 v_3 \\ &\quad + (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) x'_1 \\ &\quad + (\cos \theta_1 \sin \theta_3 - \sin \theta_1 \sin \theta_2 \cos \theta_3) x'_2 + \cos \theta_2 \cos \theta_3 x'_3. \end{aligned}$$

By assumptions  $J_1 = J_2 = J_3 = 0$ , we have  $v = x_t$ . Thus,

**Theorem 7.** *The first order compatible lift has the form:*

$$\rho^{(1)}(t, x; x_t, \theta) = \begin{bmatrix} 1 & 0 & t \\ x_t & R & x \\ 0 & 0 & 1 \end{bmatrix}.$$

The resulting one-forms  $\zeta^{(1)} = \rho^{(1)*} \mu$  will provide an invariant coframe on  $\mathcal{B}_1$ , which we name the *moving coframe of the first order*. By substituting the map  $\rho^{(1)}$  in  $\zeta^{(0)}$  and restricting to the first prolongation or jet of the curve, namely  $x = x(t)$ ,  $x_t = x'(t)$  we have,



**Theorem 8.** *The first order moving coframe is*

$$\begin{aligned}
\zeta_1^{(1)} &:= dt, \\
\zeta_2^{(1)} &:= \sin \theta_2 d\theta_1 - d\theta_3, \\
\zeta_3^{(1)} &:= \cos \theta_2 \cos \theta_3 d\theta_1 - \sin \theta_3 d\theta_2, \\
\zeta_4^{(1)} &:= \cos \theta_2 \sin \theta_3 d\theta_1 + \cos \theta_3 d\theta_2, \\
\zeta_5^{(1)} &:= \cos \theta_1 \cos \theta_2 dx'_1 - \sin \theta_1 \cos \theta_2 dx'_2 - \sin \theta_2 dx'_3, \\
\zeta_6^{(1)} &:= (\cos \theta_1 \cos \theta_2 x'_1 - \sin \theta_1 \cos \theta_2 x'_2 - \sin \theta_2 x'_3) dt \\
&\quad - \cos \theta_1 \cos \theta_2 dx_1 + \sin \theta_1 \cos \theta_2 dx_2, \\
\zeta_7^{(1)} &:= -(\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) dx'_1 \\
&\quad + (\sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3) dx'_2 - \cos \theta_2 \cos \theta_3 dx'_3, \\
\zeta_8^{(1)} &:= (\cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3) dx'_1, \\
&\quad - (\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3) dx'_2 + \cos \theta_2 \sin \theta_3 dx'_3, \\
\zeta_9^{(1)} &:= \left( (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) x'_1 + (\cos \theta_1 \sin \theta_3 \right. \\
&\quad \left. - \sin \theta_1 \sin \theta_2 \cos \theta_3) x'_2 + \cos \theta_2 \cos \theta_3 x'_3 \right) dt \\
&\quad - (\cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3) dx_1 \\
&\quad + (-\cos \theta_1 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3) dx_2 - \cos \theta_2 \cos \theta_3 dx_3, \\
\zeta_{10}^{(1)} &:= \left( (\sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3) x'_1 + (\cos \theta_1 \cos \theta_3 \right. \\
&\quad \left. - \sin \theta_1 \sin \theta_2 \sin \theta_3) x'_2 + \cos \theta_2 \sin \theta_3 x'_3 \right) dt \\
&\quad - (\cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3) dx_1 \\
&\quad + (\sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3) dx_2 - \cos \theta_2 \cos \theta_3 dx_3,
\end{aligned}$$

which is an invariant coframe on

$$\mathcal{B}_1 = \{(t, x, x_t, \theta)\} \cong J^1(\mathbb{R}; \mathbb{R}^3) \times G_3 \cong \mathbb{R}^7 \times \text{SO}(3). \quad \square$$

## 5 Second order moving coframes

By restricting  $\zeta^{(1)}$  to the second prolongation  $J^2x \times G_3$ , which is a four dimensional manifold, we have

**Theorem 9.** *The restricted (or horizontal) moving coframe forms are explicitly given by*

$$\begin{aligned}
\eta_1^{(1)} &= dt, \\
\eta_2^{(1)} &= \sin \theta_2 d\theta_1 - d\theta_3, \\
\eta_3^{(1)} &= \cos \theta_2 \cos \theta_3 d\theta_1 - \sin \theta_3 d\theta_2, \\
\eta_4^{(1)} &= \cos \theta_2 \sin \theta_3 d\theta_1 + \cos \theta_3 d\theta_2,
\end{aligned}$$

and their linear dependencies are,  $\eta_5^{(1)} = J_1\eta_1^{(1)}$ ,  $\eta_7^{(1)} = J_2\eta_1^{(1)}$ ,  $\eta_8^{(1)} = J_3\eta_1^{(1)}$ , and  $\eta_6^{(1)} = \eta_9^{(1)} = \eta_{10}^{(1)} = 0$ , where

$$\begin{aligned} J_1 &= (\cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3)x_1'' \\ &\quad - (\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3)x_2'' + \cos \theta_2 \sin \theta_3 x_3'', \\ J_2 &= -(\sin \theta_1 \sin \theta_3 + \cos \theta_3 \sin \theta_2 \cos \theta_1)x_1'' \\ &\quad + (\sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3)x_2'' - \cos \theta_2 \cos \theta_3 x_3'', \\ J_3 &= -\cos \theta_2 \cos \theta_1 x_1'' + \cos \theta_2 \sin \theta_1 x_2'' + \sin \theta_2 x_3''. \end{aligned}$$

If we assume  $(J_1, J_2, J_3) = (-a, 0, 0)$ , where the length of acceleration  $\|x_{tt}\|$  is denoted by  $a$ , then we have  $Rx_{tt} = (a, 0, 0)$ , and by simple computations, have

$$\theta_1 = -\arctan\left(\frac{x_2''}{x_1''}\right) \quad \text{and} \quad \theta_2 = \arcsin\left(\frac{x_3''}{\|x_{tt}\|}\right).$$

It can be also easily seen that  $a$  is an invariant.

Now we choose a cross section  $K = \{t = 0, x = 0, x_t = 0, \|x_{tt}\| = a, \theta = 0\}$ . By recomputing the forms  $\zeta^{(2)} = \rho^{(2)*}\mu$ , we have

**Theorem 10.** *The second order moving coframe is*

$$\begin{aligned}
\zeta_1^{(2)} &= dt, \\
\zeta_2^{(2)} &= -d\theta_3 + \frac{x_2''x_3'' dx_1''}{a(x_1''^2 + x_2''^2)} - \frac{x_1''x_3'' dx_2''}{a(x_1''^2 + x_2''^2)}, \\
\zeta_3^{(2)} &= \frac{x_1''x_3'' \sin \theta_3 + ax_2'' \cos \theta_3}{a^2 \sqrt{x_1''^2 + x_2''^2}} dx_1'' - \frac{x_2''x_3'' \sin \theta_3 + ax_1'' \cos \theta_3}{a^2 \sqrt{x_1''^2 + x_2''^2}} dx_2'' \\
&\quad + \frac{\sin \theta_3 \sqrt{x_1''^2 + x_2''^2}}{a^2} dx_3'', \\
\zeta_4^{(2)} &= -\frac{x_1''x_3'' \cos \theta_3 + ax_2'' \sin \theta_3}{a^2 \sqrt{x_1''^2 + x_2''^2}} dx_1'' - \frac{x_2''x_3'' \cos \theta_3 + ax_1'' \sin \theta_3}{a^2 \sqrt{x_1''^2 + x_2''^2}} dx_2'' \\
&\quad + \frac{\cos \theta_3 \sqrt{x_1''^2 + x_2''^2}}{a^2} dx_3'', \\
\zeta_5^{(2)} &= -\frac{x_1''}{a} dx_1' - \frac{x_2''}{a} dx_2' - \frac{x_3''}{a} dx_3', \\
\zeta_6^{(2)} &= \frac{x_1'x_1'' + x_2'x_2'' + x_3'x_3''}{a} dt - \frac{x_1''}{a} dx_1 - \frac{x_2''}{a} dx_2 - \frac{x_3''}{a} dx_3, \\
\zeta_7^{(2)} &= \frac{ax_2'' \sin \theta_3 - x_1''x_3'' \cos \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_1 - \frac{ax_1'' \sin \theta_3 + x_2''x_3'' \cos \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_2 \\
&\quad - \frac{\sqrt{x_1''^2 + x_2''^2} \cos \theta_3}{a} dx_3, \\
\zeta_8^{(2)} &= \frac{ax_2'' \cos \theta_3 - x_1''x_3'' \sin \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_1 + \frac{-ax_1'' \cos \theta_3 + x_2''x_3'' \sin \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_2 \\
&\quad + \frac{\sqrt{x_1''^2 + x_2''^2} \sin \theta_3}{a} dx_3, \\
\zeta_9^{(2)} &= \frac{a(x_2'x_1'' - x_1'x_2'') \cos \theta_3 - (x_2'x_3''x_2'' + x_1'x_3''x_1'' + x_3'x_1''^2 + x_3'x_2''^2) \sin \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dt \\
&\quad + \frac{ax_2'' \cos \theta_3 + x_1''x_3'' \sin \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_1 - \frac{ax_1'' \cos \theta_3 - x_2''x_3'' \sin \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_2 \\
&\quad + \frac{\sin \theta_3 \sqrt{x_1''^2 + x_2''^2}}{a} dx_3, \\
\zeta_{10}^{(2)} &= \frac{a(x_2'x_1'' - x_1'x_2'') \sin \theta_3 + (x_2'x_3''x_2'' + x_1'x_3''x_1'' + x_3'x_1''^2 + x_3'x_2''^2) \cos \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dt \\
&\quad + \frac{ax_2'' \sin \theta_3 - x_1''x_3'' \cos \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_1 - \frac{ax_1'' \sin \theta_3 + x_2''x_3'' \cos \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_2 \\
&\quad - \frac{\cos \theta_3 \sqrt{x_1''^2 + x_2''^2}}{a} dx_3.
\end{aligned}$$

For any constant  $a > 0$ , these forms serve as a coframe on

$$\mathcal{B}_2 = \left\{ (t, x, x_t, x_{tt}, \theta) \in \mathbb{R}^{10} \times \text{SO}(3) \mid \theta_1 = -\arctan\left(\frac{x_2''}{x_1''}\right), \theta_2 = \arcsin\left(\frac{x_3''}{\|x_{tt}\|}\right), \|x_{tt}\| = a \right\}$$

## 6 Third order moving coframes

By restricting to the third prolongation  $J^3\mathcal{B}_2$  which is a 2-dimensional manifold, we have,

**Theorem 11.** *The restricted (or horizontal) moving coframe forms are explicitly given by  $\eta_1^{(2)} = dt$ ,*

$$\eta_2^{(2)} = \frac{x_3''(x_1'x_2'' - x_2'x_1'')}{a(x_1''^2 + x_2''^2)} \eta_1^{(2)} - d\theta_3,$$

their linear dependencies in this step are  $\eta_3^{(2)} = J_1\eta_1^{(2)}$ ,  $\eta_4^{(2)} = J_2\eta_1^{(2)}$ ,  $\eta_5^{(2)} = -a\eta_1^{(2)}$  and  $\eta_6^{(2)} = \eta_7^{(2)} = \eta_8^{(2)} = \eta_9^{(2)} = \eta_{10}^{(2)} = 0$ ; Where

$$\begin{aligned} J_1 &= \frac{1}{a^2\sqrt{x_1''^2 + x_2''^2}} \left\{ a(x_1^{(3)}x_2'' - x_2^{(3)}x_1'') \cos \theta_3 \right. \\ &\quad \left. + ((x_1''x_1^{(3)} + x_2^{(3)}x_2'')x_3'' - (x_1''^2 + x_2''^2)x_3^{(3)}) \sin \theta_3 \right\}, \\ J_2 &= \frac{1}{a^2\sqrt{x_1''^2 + x_2''^2}} \left\{ a(x_1^{(3)}x_2'' - x_2^{(3)}x_1'') \sin \theta_3 \right. \\ &\quad \left. + ((x_1''^2 + x_2''^2)x_3^{(3)} - (x_1''x_1^{(3)} + x_2^{(3)}x_2'')x_3'') \cos \theta_3 \right\}. \end{aligned}$$

If we assume  $J_1 = 0$ , then we find that  $J_2 = \frac{\|x_{tt} \times x_{ttt}\|}{a^2}$  and

$$\theta_3 = \arctan\left(\frac{a(x_1^{(3)}x_2'' - x_2^{(3)}x_1'')}{(x_1''^2 + x_2''^2)x_3^{(3)} - (x_1''x_1^{(3)} + x_2^{(3)}x_2'')x_3''}\right).$$

thus,

**Theorem 12.** *The most general third order compatible lift has the form*

$$\rho^{(3)}(t, x; x_t, x_{tt}, x_{ttt}) = \begin{bmatrix} 1 & 0 & 0 & 0 & t \\ x_t & \frac{x_{tt}}{\|x_{tt}\|} & \frac{x_{tt} \times x_{ttt}}{\|x_{tt} \times x_{ttt}\|} & \frac{x_{tt} \times (x_{tt} \times x_{ttt})}{\|x_{tt} \times (x_{tt} \times x_{ttt})\|} & x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now by substitution, we have

**Theorem 13.** *The third order moving coframe  $\zeta^{(3)} = \rho^{(3)*}\mu$  is*

$$\begin{aligned}
\zeta_1^{(3)} &= dt, \\
\zeta_2^{(3)} &= \frac{\mathbf{x}_{tt} \times \mathbf{x}_{ttt}}{a \|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|^2} \cdot \left( (\mathbf{x}_{tt} \cdot \mathbf{x}_{ttt}) d\mathbf{x}_{tt} + a^2 d\mathbf{x}_{ttt} \right), \\
\zeta_3^{(3)} &= -\frac{\mathbf{x}_{tt} \cdot \mathbf{x}_{ttt}}{a \|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} (\mathbf{x}_{tt} \times \mathbf{x}_{ttt}) \cdot d\mathbf{x}_{tt}, \\
\zeta_4^{(3)} &= -\frac{\mathbf{x}_{tt} \times (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{a^2 \|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}_{tt}, \\
\zeta_5^{(3)} &= -\frac{1}{a} \mathbf{x}_{tt} \cdot d\mathbf{x}_t, \\
\zeta_6^{(3)} &= \frac{\mathbf{x}_t \cdot \mathbf{x}_{tt}}{a} dt - \frac{1}{a} \mathbf{x}_{tt} \cdot d\mathbf{x}, \\
\zeta_7^{(3)} &= \frac{\mathbf{x}_{tt} \times (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{a \|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}_t, \\
\zeta_8^{(3)} &= \frac{\mathbf{x}_{tt} \times \mathbf{x}_{ttt}}{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}_t, \\
\zeta_9^{(3)} &= -\frac{\mathbf{x}_t \cdot (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} dt + \frac{\mathbf{x}_{tt} \times \mathbf{x}_{ttt}}{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}, \\
\zeta_{10}^{(3)} &= \frac{(\mathbf{x}_t \times \mathbf{x}_{tt}) \cdot (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{a \|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} dt - \frac{\mathbf{x}_{tt} \times (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{a \|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}.
\end{aligned}$$

For any constant  $a > 0$ , these forms serve as a coframe on

$$\begin{aligned}
\mathcal{B}_3 &= \left\{ (t, \mathbf{x}, \mathbf{x}_t, \mathbf{x}_{tt}, \theta) \in \mathbb{R}^{10} \times \text{SO}(3) \mid \right. \\
&\quad \theta_1 = -\arctan\left(\frac{x_2''}{x_1''}\right), \theta_2 = \arcsin\left(\frac{x_3''}{\|\mathbf{x}_{tt}\|}\right), \|\mathbf{x}_{tt}\| = a, \\
&\quad \left. \theta_3 = \arctan\left(\frac{a(x_1^{(3)}x_2'' - x_2^{(3)}x_1'')}{(x_1''^2 + x_2''^2)x_3^{(3)} - (x_1'x_1^{(3)} + x_2'x_2^{(3)})x_3''}\right) \right\}.
\end{aligned}$$

**Theorem 14.** *The restricted (or horizontal) moving coframe forms are explicitly given by  $\eta_1^{(3)} = dt$ ,  $\eta_4^{(3)} = \frac{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|}{a^2} \eta_1^{(3)}$ ,  $\eta_3^{(3)} = \eta_5^{(3)} = \eta_6^{(3)} = \eta_7^{(3)} = \eta_9^{(3)} = \eta_{10}^{(3)} = 0$ ,  $\eta_8^{(3)} = -a\eta_1^{(3)}$ , and  $\eta_2^{(3)} = J\eta_1^{(3)}$ , where  $J = a((\mathbf{x}_{tt} \times \mathbf{x}_{ttt}) \cdot \mathbf{x}_{ttt})/\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|^2$ .*

**Theorem 15.**  $\frac{d}{dt}$  is a differential operator and the functions  $a_1 := \|\mathbf{x}_{tt}\|$ ,  $a_2 := \|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|$  and  $a_3 := (\mathbf{x}_{tt} \times \mathbf{x}_{ttt}) \cdot \mathbf{x}_{ttt}$  are differential invariants.

## 7 Dimensional considerations

In this section, we use the conventions of chapter 5 of [8].

If we use the coordinates  $(t, x, x_t, x_{tt}, \dots, x^{(n)})$  for  $J^n \mathbb{R}^4$ , then the prolonged group action  $\text{SGal}(n)$  on  $J^n \mathbb{R}^4$  can be written as  $\bar{t} = t + s$ ,  $\bar{x} = Rx + tv + y$ ,  $\bar{x}_t = Rx_t + v$ , and  $\bar{x}^{(n)} = Rx^{(n)}$  for  $n \geq 2$ .

It is recommended that the dimension of  $J^n \mathbb{R}^4$  is  $p+q^n = 3n+4$ , and the dimension of  $\text{SGal}(n)$  is 10.

**Theorem 16.** *The following functions are differential invariants:*

- 1)  $I_n := \|x^{(n)}\|$  for  $n \geq 2$ .
- 2)  $J_{n,m} := x^{(n)} \cdot x^{(m)}$  for  $n > m \geq 2$ .
- 3)  $K_{n,m} := \|x^{(n)} \times x^{(m)}\|$  for  $n > m \geq 2$ .
- 4)  $L_{l,n,m} := (x^{(l)} \times x^{(n)}) \cdot x^{(m)}$  for  $l > n > m \geq 2$ .

*Proof.* If  $n, m \geq 2$ , then since  $\bar{x}^{(n)} = Rx^{(n)}$ ,  $\bar{x}^{(m)} = Rx^{(m)}$  and  $R \in \text{SO}(3)$ , therefore  $\bar{x}^{(n)} \cdot \bar{x}^{(m)} = x^{(n)} \cdot x^{(m)}$ ; hence  $I_{n,m}$  is an invariant.

By (1), (2) and formulas  $\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \cdot v)^2$ , we find that  $K_{n,m} = I_n I_m - J_{n,m}^2$  is an invariant.

Since  $(u_1 \times u_2) \cdot u_3 = \sqrt{\det(u_i \cdot u_j)}$ , then  $L_{l,n,m}$  is a function of  $J_{n,m}$ 's, and this complete the proof.  $\square$

By usual computations, we find that the maximal dimension of prolonged action are:  $s_0 = 4$ ,  $s_1 = 7$ ,  $s_2 = 9$ ,  $s_n = 10$  for  $n \geq 3$ . Therefore, the order of this group action is  $s = 3$ .

Therefore, the  $i_n$  functionally independent differential invariants of order at most  $n$  are:  $i_0 = i_1 = 0$ ,  $i_2 = 1$  and  $i_n = 3(n-2)$  for  $n \geq 3$ .

By the theorem 5.31 of [8], we have

**Theorem 17.** *The complete system of 3<sup>rd</sup> order differential invariants of special Galilean group action are  $\|x_{tt}\|$ ,  $\|x_{ttt}\|$  and  $x_{tt} \cdot x_{ttt}$ . Locally, every 3<sup>rd</sup> order differential invariant of  $\text{SGal}(3)$  can be written as a function of these differential invariants.*  $\square$

**Corollary 1.**  $a_1 = I_2$ ,  $a_2 = J_{3,2}$ ,  $a_3 = L_{4,3,2}/K_{3,2}^2$ .  $\square$

**Theorem 18.** *Every differential invariant of special Galilean group action is a function of  $a := \|x_{tt}\|$ ,  $b := \|x_{ttt}\|$  and their derivatives with respect to  $t$ .*

*Proof.* According to Theorem 17, it is enough to show that  $x_{tt} \cdot x_{ttt}$  can be written as a function of  $\|x_{tt}\|$  and  $\|x_{ttt}\|$ . But  $\frac{1}{2} \frac{d}{dt} \|x_{tt}\|^2 = x_{tt} \cdot x_{ttt}$ .  $\square$

## 8 Conclusions

The necessary condition for the local special Galilean equivalence of two given motions, is that the corresponding invariants are the same. These produce a large amount of necessary conditions.

For the sufficiency condition of the equivalence for coframes  $\zeta$ , we can rewrite 2-forms  $d\zeta_i$  in terms of wedge products of the 1-forms  $\zeta_i$ . These produce the *structure functions*, which are our *original invariants*, by differentiating, we have *derived invariants*, whose functional interrelationships provide a necessary condition for equivalence. By introducing the *structure invariants*, we can define the components of the *structure map*. The  $s$ -order *classifying space* and the fully regularity condition on  $s$ -order *structure map* leads us to the definition of  $s$ -order *classifying manifold*  $\mathcal{C}^{(s)}$  due to chapter 8 in [8]. In view of the Proposition 8.11 in [8], the necessary conditions for the (local) equivalence of coframes state that for each  $s \geq 0$ , their  $s$ -order classifying manifolds overlapp. The full regularity conditions provide that these necessity conditions are also sufficient.

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