The fractional transport equation: an analytical solution and a spectral approximation by Chebyshev polynomials

Abdelouahab Kadem

Abstract. In this paper, we show that the fractional transport equation can be reduced to a fractional linear differential equation system by using Chebyshev polynomials. We give the analytical solution of this system followed by the spectral approximation.

M.S.C. 2000: 65D32, 82D75, 40A10, 41A50.
Key words: Linear transport equation, isotropic scattering, Chebyshev spectral method, discrete-ordinates method.

1 Introduction

The fractional derivatives technique has been widely employed for solving linear differential and integral equations, among them is cited the diffusion problems in curvilinear coordinates [7, 12].

In our recent work we have presented a new method for solving a steady fractional transport equation in two dimensional case using the orthogonal properties of Chebyshev polynomials and Walsh functions, in ordinary case a new approximation for solving the one dimensional transport equation analytically, have been reported where we are using the Chebyshev polynomials combined with the Sumudu transform [4]. The approach is based on expansion of the angular flux in a truncated series of Chebyshev polynomials in the angular variable. By replacing this development in the transport equation, this which will result a first-order linear differential system is solved for the spatial function coefficients by application of the Sumudu transform technique [11]. The inversion of the transformed coefficients is obtained using Trzaska's method [10] and the Heaviside expansion technique.

In order to obtain a new rule for the calculate the matrix exponential which arises in the formal solution of algebraic systems of differential equation without using the Sumudu transform and Trzaska's method for solving the first-order fractional linear differential system, we fractionalize the one dimensional integro-differential equation and we try to convert it into a system of fractional differential equation (**FDE**). The main characteristic of using this technique is that reduces this problems to those of

Applied Sciences, Vol.11, 2009, pp. 78-90.

solving a system of algebraic equations, thus greatly simplifying the problem and making it computational plausible on the other hand that permit us to solve some of the particular cases and then we can check that the solution is close to the dynamics of some anomalous processes, the other aspect of this technique, it allows us to establish a fractional derivative which performs the same mapping of a given linear operator, it becomes to use the Riemann-Liouville definition for fractional derivatives and considerate the ordinary model and look that in the limit of some situations where the ordinary model do not work fine it is necessary to introduce such fractional operators in the model so we solve the problem.

The paper has been organized as follows. Section 2 contains preliminaries, and Section 3 describes how to convert a transport equation into FDE in Section 4 we report a specification application of the method.

Let us consider the following mono-energetic $3-D$ transport equation:

$$\underline{\Omega}.\underline{\nabla}(\underline{r},\underline{\Omega}) + \sigma_t \Psi(\underline{r},\underline{\Omega}) = \int_{4\pi} \sigma_s(\underline{\Omega},\underline{\Omega'})\Psi(\underline{r},\underline{\Omega'})d\Omega' + \frac{1}{4\pi}Q(\underline{r}) \tag{1.1}$$

$$\underline{\Omega} = (\eta, \xi) \quad \text{angular variable,} \tag{1.2}$$

and

$$\sigma_s(\mu_0) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} \sigma_{sk} P_k(\mu_0) \quad \text{differential cattering cross section} \tag{1.3}$$

with $\mu_0 = \underline{\Omega}.\underline{\Omega'}$ and $P_k =$ the k^{th} Legendre polynomial.

2 Preliminaries

We enlist some definitions and basic results [5, 6, 9]

Definition 2.1. A real function $f(x), x > 0$ is said to be in the space $C_{\alpha,\alpha\in R}$ if there exists a real number $p(> \alpha)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) = C\,[0, \infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$.

Definition 2.2. A function $f(x), x > 0$ is said to be in space C_α^m, $m \in N \cup \{0\}$, if $f_{(m)} \in C_\alpha$.

Definition 2.3. The (left sided) Riemann-Liouville fractional integral of order $\mu > 0$, of a function $f \in C_\alpha, \alpha \geq 1$ is defined as:

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau)d\tau, \mu > 0, t > 0, \tag{2.1}$$

$$I^0 f(t) = f(t)$$

Definition 2.4. The (left sided) Riemann-Liouville fractional derivative of f, $f \in C_{-1}^m$, $m \in N \cup \{0\}$ of order $\alpha > 0$, is defined as:

$$D^\mu f(t) = \frac{d^m}{dt^m} I^{m-\mu} f(t), m-1 < \mu \leq m, m \in N. \tag{2.2}$$

Definition 2.5. The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^{m}$, $m \in N \cup \{0\}$ of order $\alpha > 0$, is defined as:

$$D_c^{\mu} f(t) = \begin{cases} \left[I^{m-\mu} f^{(m)}(t)\right], m-1 < \mu \leq m, m \in N, \\ \frac{d^m}{dt^m} f(t) \qquad \mu = m. \end{cases} \tag{2.3}$$

Note that
(i) $I^{\mu} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} t^{\gamma+\mu}, \qquad \mu > 0,\ \gamma > -1,\ t > 0.$
(ii) $I^{\mu} D_c^{\mu} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{t^k}{k!},\ m-1 < \mu \leq m,\ m \in N.$
(iii) $D_c^{\mu} f(t) = D^{\mu} \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{t^k}{k!} \right),\ m-1 < \mu \leq m,\ m \in N.$
(iv) $D^{\beta} I^{\alpha} f(t) = \begin{cases} I^{\alpha-\beta} f(t) & \text{if} \quad \alpha > \beta, \\ f(t) & \text{if} \quad \alpha = \beta, \\ D^{\beta-\alpha} f(t) & \text{if} \quad \alpha < \beta, \end{cases}$
(v) $D_c^{\alpha} D^m f(t) = D^{\alpha+m} f(t),\ m = 0, 1, 2, ..., n-1 < \alpha < n.$
From here on, we will use $C_{\gamma}([a, b])$ $(\gamma \in R)$ to denote the Banach space

$$C_{\gamma}([a, b]) = \left\{ g(x) \in C((a, b]) : \| g \|_{C_{\gamma}} = \| (x-a)^{\gamma} g(x) \|_{C} < \infty \right\}. \tag{2.4}$$

In particular, $C_0([a, b])$ represents the space of continuous functions in $[a, b]$, that is, $C([a, b])$.

3 Planar Geometry

We consider a planar-geometry problem with spatial variation only in the $x-$direction:

$$Q(\underline{r}) = q(x), \tag{3.1}$$

$$\Psi(\underline{r}, \underline{\Omega}) = \frac{1}{2\pi} \Psi(x, \mu) \tag{3.2}$$

equation (1) simplifies to

$$\mu \frac{\partial \Psi}{\partial x}(x, \mu) + \sigma_t \Psi(x, \mu) = \int_{-1}^{1} \sigma_s(\mu, \mu') \Psi(x, \mu') d\mu' + \frac{q(x)}{2}, \tag{3.3}$$

with

$$\sigma_s(\mu, \mu') = \sum_{k=0}^{\infty} \frac{2k+1}{2} \sigma_{sk} P_k(\mu) P_k(\mu'). \tag{3.4}$$

So we consider equation (3.3) with $0 \leq x \leq a$ and $-1 \leq \mu \leq 1$, and subject to the boundary conditions

$$\Psi(x = 0, -\mu) = f(\mu), \tag{3.5}$$

and

$$\Psi(x = a, \mu) = 0, \tag{3.6}$$

where $f(\mu)$ is the prescribed incident flux at $x = 0$; $\Psi(x, \mu)$ is the angular flux in the μ direction; σ_t is the total cross section; σ_{sl}, with $l = 0, 1, ..., L$ are the components of the differential scattering cross section, and $P_k(\mu)$ are the Legendre polynomials of degree k. Now we fractionalize the equation (3.3) with the same boundary conditions (3.5) and (3.6)

$$\mu \frac{\partial^\beta \Psi}{\partial x^\beta}(x, \mu) + \sigma_t \Psi(x, \mu) = \int_{-1}^{1} \sigma_s(\mu, \mu')\Psi(x, \mu')d\mu' + \frac{q(x)}{2}, \tag{3.7}$$

with $0 < \beta \leq 1$.

Theorem 3.1. *Consider the integro−differential equation* (3.7) *subject to the boundary conditions* (3.5) *and* (3.6), *then the function* $\Psi(x, \mu)$ *satisfy the follow first − order linear differential equation system for the spatial component* $Y_n(x)$

$$\sum_{n=0}^{N} \alpha^1_{n,m} Y_n^{(\beta)}(x) + \frac{\sigma_t \pi}{2 - \delta_{m,0}} Y_m(x) = \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{m,l} \sum_{n=0}^{N} \alpha^3_{n,l} Y_n(x) + \frac{q(x)}{2}$$

where

$$\alpha^1_{n,m} := \int_{-1}^{1} \mu T_n(\mu) \frac{T_m(\mu)}{\sqrt{1-\mu^2}} d\mu,$$

$$\alpha^2_{n,l} := \int_{-1}^{1} T_n(\mu) P_l(\mu) d\mu,$$

$$\alpha^3_{n,l} := \int_{-1}^{1} \frac{T_n(\mu) P_l(\mu)}{\sqrt{1-\mu^2}} d\mu,$$

and $Y_m(x)$ *are the coefficients of the expansion of the* $\Psi(x, \mu)$.

To prepare for the proof of the theorem we need the following result

Proposition 3.2. *Let*

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$$

and

$$P_{l+1}(x) = 2xP_l(x) - P_{l-1}(x) - [xP_l(x) - P_{l-1}(x)] / (l+1)$$

the recurrence relations for the Chebyshev and the Legendre polynomials respectively we have for $l > 2$ *and* $k = 2, 3$

$$\alpha^k_{n,l+1} := \frac{2l+1}{2l+2} \left[\alpha^k_{n+1,l} + \alpha^k_{n-1,l}\right] - \frac{l}{l+1} \alpha^k_{n,j-1}$$

Hence, in particular for $l = 0$ *and* 1 *the coefficients* $\alpha^2_{n,l}$ *and* $\alpha^3_{n,l}$ *assume the values*

$$\alpha^2_{n,l} = \begin{cases} 0 & \text{if} \quad n+l \quad \text{odd}, \\ \frac{2}{(1+l)^2 - n^2} & \text{if} \quad n+l \quad \text{even}, \end{cases}$$

and

$$\alpha^3_{n,l} = \frac{\pi \delta_{n,l}}{2 - \delta_{l,0}}$$

Proof. It easy to see that

$$\alpha^1_{n,m} = \frac{\pi \delta_{|n-m|}}{2(2-\delta_{n+m,1})}$$

For $k = 2$ by the multiplication of the Chebyshev and the Legendre recurrence formulas we have

$$\frac{2l+1}{2l+2}\left[P_l(\mu)T_{n+1}(\mu) + P_l(\mu)T_{n-1}(\mu)\right] - \frac{l}{2\mu\,(l+1)}P_{l-1}(\mu)\left[T_{n+1}(\mu) + T_{n-1}(\mu)\right]$$

it is known that

$$T_{n+1}(\mu) + T_{n-1}(\mu) = 2\mu T_n(\mu)$$

after doing some algebraic manipulations and integrating over $\mu \in [-1,1]$ on the resulting equation we get

$$\alpha^2_{n,l+1} = \frac{2l+1}{2l+2}\left[\alpha^2_{n+1,l} + \alpha^2_{n-1,l}\right] - \frac{l}{l+1}\alpha^2_{n,j-1}$$

The case $k = 3$ is treated similarly but in this case we multiply the resulting expression by $\frac{1}{\sqrt{1-\mu^2}}$ and integrate over $\mu \in [-1,1]$ we get the desired result.

Now we give a proof of Theorem 3.1.

Proof. Expanding the angular flux in the μ variable in terms of the Chebyshev polynomials leads to $\Psi(x,\mu) = \sum_{n=0}^{N} \frac{Y_n(x)T_n(\mu)}{\sqrt{1-\mu^2}}$ with

$N = 0, 2, 4, ...$, where the expansions coefficients $Y_n(x)$ should be determined. Here $T_n(\mu)$ are the well known Chebyshev polynomials of order n which are orthogonal in the interval $[-1,1]$ with respect to the weight $w(t) = 1/\sqrt{1-t^2}$. After replacing this ansatz into equation (3.7) it turns out

$$\sum_{n=0}^{N}\left\{\mu Y_n^{(\beta)}(x) + \sigma_t Y_n(x)\right\}\frac{T_n(\mu)}{\sqrt{1-\mu^2}} =$$

$$\sum_{l=0}^{L}\frac{2l+1}{2}\sigma_{sl}P_l(\mu)\sum_{n=0}^{N}Y_n(x)\int_{-1}^{1}P_l(\mu')\frac{T_n(\mu')}{\sqrt{1-\mu'^2}}d\mu' + \frac{q(x)}{2} \tag{3.8}$$

using the orthogonality of the Chebyshev polynomials, multiply the equation (3.8) by $T_m(\mu)$, considering $m = 0, 1, ..., N$, and integrated in the μ variable in the interval $[-1,1]$. Thus we get the following first-order linear differential equation system for the spatial component $Y_n(x)$

$$\sum_{n=0}^{N}\alpha^1_{n,m}Y_n^{(\beta)}(x) + \frac{\sigma_t \pi}{2-\delta_{m,0}}Y_m(x) = \sum_{l=0}^{L}\frac{2l+1}{2}\sigma_{sl}\alpha^2_{m,l}\sum_{n=0}^{N}\alpha^3_{n,l}Y_n(x) + \frac{q(x)}{2} \tag{3.9}$$

where

$$\alpha^1_{n,m} = \int_{-1}^{1}\mu T_n(\mu)\frac{T_m(\mu)}{\sqrt{1-\mu^2}}d\mu, \tag{3.10}$$

$$\alpha_{n,l}^{2} = \int_{-1}^{1} T_n(\mu) P_l(\mu) d\mu, \tag{3.11}$$

$$\alpha_{n,l}^{3} = \int_{-1}^{1} \frac{T_n(\mu) P_l(\mu)}{\sqrt{1-\mu^2}} d\mu, \tag{3.12}$$

with $\delta_{n,m}$ denoting the delta of Kronecker. Here the coefficients $\alpha_{n,l}^{2}$ and $\alpha_{n,l}^{3}$ are evaluated by the multiplication of the Chebyshev and Legendre recurrence formulas and integration of the resulting equation (See proposition 3.2).

In the next step we solve the β-order linear differential equation system (3.9), we rewrite this equation in matrix form

$$A.Y^{(\beta)}(x) + BY(x) = C(x) \tag{3.13}$$

where $Y^{(\beta)}$ is the Caputo fractional derivative of order β, with $0 < \beta \leq 1$,

$Y(x) = Col.\,[Y_0(x), Y_1(x), ..., Y_N(x)]$ and A is a real square matrix of order $N+1$, that is $A \in M_{N+1}(R)$, and $B \in C_{1-\beta}((0,x])$, with the components

$$(A)_{i,j} = \alpha_{i-1,j-1}^{1}, \tag{3.14}$$

$$(B)_{i,j} = \frac{\pi \sigma_t}{2 - \delta_{1,j}} \delta_{i,j} - \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha_{i-1,l}^{2} \sum_{n=0}^{N} \alpha_{j-1,l}^{3} \tag{3.15}$$

where each component of B belongs to space $C_{1-\beta}((0,x])$, and

$$C(x) = \frac{q(x)}{2} = Col.\,[C_0(x), C_1(x), ..., C_N(x)]\,. \tag{3.16}$$

We rewrite the expression (3.14) as

$$Y^{(\beta)}(x) + A^{-1}BY(x) = A^{-1}C(x) \tag{3.17}$$

We notice that the general solution to (3.18) where $A^{-1}B \in M_{N+1}(R)$, and $A^{-1}C \in \mathcal{C}_{1-\beta}((0,x])$, is given by [1]

$$Y(x) = e_{\beta}^{A^{-1}Bx} H + \int_0^x e_{\beta}^{A^{-1}B(x-\xi)} A^{-1} C(\xi) d\xi, \tag{3.18}$$

where H is an arbitrary constant matrix, and the β-exponential function $e_{\beta}^{A^{-1}Bx}$ is defined by

$$e_{\beta}^{A^{-1}Bx} = x^{\beta-1} \sum_{k=0}^{\infty} (A^{-1}B)^k \frac{x^{k\beta}}{\Gamma\left[(k+1)\beta\right]} \tag{3.19}$$

Moreover, it is easy to see that the function $e_{\beta}^{A^{-1}B}$ satisfies the following properties [1]:

(i) If $\parallel A^{-1}B \parallel = \max_{i,j} \mid A_{i,j} B_{i,j} \mid$, where $A_{i,j}$ and $B_{i,j}$ are the components of matrix A and B respectively then

$\parallel e_{\beta}^{A^{-1}B} \parallel \leq \sum_{k=0}^{\infty} \parallel A^{-1}B \parallel^{k} \frac{x^{(k+1)\beta-1}}{\Gamma[(k+1)\beta]}$ $(x > 0)$,

(ii) $e_{\beta}^{A^{-1}Bx} e_{\beta}^{Kx} \neq e_{\beta}^{(A^{-1}B+K)x}$ $(\beta \neq 1)$,

(iii) $D^{\beta} e_{\beta}^{A^{-1}Bx} = (A^{-1}B) e_{\beta}^{A^{-1}Bx}$,

where $A, B, K \in M_n(R)$ and $\beta \in (0,1]$.

4 Specific application of the method

Consider the two-dimensional linear steady state transport equation given by

$$\mu\frac{\partial^\beta}{\partial x^\beta}\Psi(\mathbf{x},\mu,\phi)+\sqrt{1-\mu^2}\cos\phi\frac{\partial^\beta}{\partial y^\beta}\Psi(\mathbf{x},\mu,\phi)+\sigma_t\Psi(\mathbf{x},\mu,\phi)$$

$$=\int_{-1}^{1}\int_0^{2\pi}\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)\Psi(\mathbf{x},\mu^{'},\phi^{'})d\phi^{'}d\mu^{'}+S(\mathbf{x},\mu,\phi) \tag{4.1}$$

in the rectangular domain $\Omega=\{\mathbf{x}:=(x,y)\mathbf{:}\quad -1\leq x\leq 1,\quad -1\leq y\leq 1\}$, $0<\beta\leq 1$, and the direction in

$D=\{(\mu,\theta):\quad -1\leq\mu\leq 1,\quad 0\leq\theta\leq 2\pi\}$. Here $\Psi(\mathbf{x},\mu,\phi)$ is the angular flux, σ_t and σ_s denote the total and the differential cross section, respectively, $\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)$ describes the scattering from an assumed pre-collision angular coordinates $(\mu^{'},\theta^{'})$ to a post-collision coordinates (μ,θ) and S is the source term.

Given the functions $f_1(y,\mu,\phi)$ and $f_2(x,\mu,\phi)$, describing the incident flux, we seek for a solution of (4.1) subject to the following boundary conditions:

For $0\leq\theta\leq 2\pi$, let

$$\Psi(x=\pm1,y,\mu,\theta)=\begin{cases} f_1(y,\mu,\phi), & x=-1,\ \ 0<\mu\leq 1,\\ 0, & x=1,\ -1\leq\mu<0.\end{cases} \tag{4.2}$$

For $-1<\mu<1$, let

$$\Psi(x,y=\pm1,\mu,\theta)=\begin{cases} f_2(y,\mu,\phi), & y=-1,\ 0<\cos\theta\leq 1,\\ 0, & y=1,\ \ -1\leq\cos\theta<0.\end{cases} \tag{4.3}$$

Theorem 4.1. *Consider the* $integro-differential$ *equation* (4.1) *under the boundary conditions* (4.2) *and* (4.3), *then the* $function$ $\Psi(\mathbf{x},\mu,\theta)$ $satisfies$ the $following$ $first-order$ $fractional$ $linear$ $differential$ $equation$ for the $spatial$ *component* $\Psi_i(x)$

$$\mu\frac{\partial^\beta}{\partial x^\beta}\Psi_k(x,\mu,\theta)+\sigma_t\Psi_k(x,\mu,\theta)$$

$$\int_{-1}^{1}\int_0^{2\pi}\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)\Psi_k(x,\mu^{'},\phi^{'})d\theta^{'}d\mu^{'}+G_k(x,\mu,\theta)$$

Proof. Expanding the angular flux $\Psi(\mathbf{x},\mu,\theta)$ in terms of the Chebyshev polynomials in the y variable, leads to

$$\Psi(\mathbf{x},\mu,\theta)=\sum_{i=0}^{I}\Psi_i(x,\mu,\theta)T_i(y). \tag{4.4}$$

Below we determine the first component, i.e., $\Psi_0(x,\mu,\theta)$ explicitly, whereas the other components, $\Psi_i(x,\mu,\theta)$, $i=1,...I$, will appear as the unknowns in I one dimensional transport equations. We start to determine $\Psi_0(x,\mu,\theta)$, by inserting (4.4) into the boundary conditions (4.3) at $y=\pm 1$, to find that:

$$\Psi_0(x,\mu,\theta) = f_2(x,\mu,\phi) - \sum_{i=1}^{I} (-1)^i \Psi_i(x,\mu,\theta), 0 < \cos\theta \le 1, \tag{4.5}$$

$$\Psi_0(x,\mu,\theta) = -\sum_{i=1}^{I} \Psi_i(x,\mu,\theta), \quad -1 \le \cos\theta < 0. \tag{4.6}$$

where $-1 \le x \le 1$, $-1 < \mu < 1$, and we have used the fact that for the Chebyshev polynomials $T_0(x) \equiv 0$, $T_i(1) \equiv 1$ and $T_i(-1) \equiv (-1)^i$.

If we now insert Ψ from (4.4) into (4.1), multiply the resulting equation by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k = 1, ..., I$, and integrate over y we find that the components $\Psi_k(x,\mu,\theta)$, $k = 1, ..., I$, satisfy the following I one-dimensional equations:

$$\mu \frac{\partial^\beta}{\partial x^\beta} \Psi_k(x,\mu,\theta) + \sigma_t \Psi_k(x,\mu,\theta)$$

$$\int_{-1}^{1} \int_{0}^{2\pi} \sigma_s(\mu^{'}, \phi^{'} \to \mu, \phi) \Psi_k(x, \mu^{'}, \phi^{'}) d\theta^{'} d\mu^{'} + G_k(x,\mu,\theta) \tag{4.7}$$

The same procedure with the boundary condition (4.2) at $x=-1$, and (4.4) yields

$$\Psi(-1,y,\mu,\theta) = f_1(y,\mu,\phi) = \sum_{i=0}^{I} \Psi_i(-1,\mu,\theta) T_i(y). \tag{4.8}$$

Now multiply (4.8) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k = 1, ..., I$, and integrate over y we find that

$$\Psi_k(-1,\mu,\theta) = \frac{2}{\pi} \int_{-1}^{1} f_1(y;\mu,\theta) \frac{T_k(y)}{\sqrt{1-y^2}} dy. \tag{4.9}$$

Similarly, (note the sign of μ below), the boundary condition at $x=1$ is written as

$$\sum_{i=0}^{I} \Psi_i(1,-\mu,\theta) T_i(y) = 0 \quad 0 < \mu \le 1. \tag{4.10}$$

Multiplying (4.10) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k = 1, ..., I$ and integrating over y, we get

$$\Psi_k(1,-\mu,\theta) = 0, 0 < \mu \le 1, 0 \le \theta \le 2\pi. \tag{4.11}$$

We can easily check that G_k in (4.7) is written as

$$G_k(x,\mu,\theta) = S_k(x,\mu,\theta) - \sqrt{1-\mu^2} \cos\theta \sum_{i=k+1}^{I} A_i^k \Psi_k(x,\mu,\theta) \tag{4.12}$$

where

$$A_i^k = \frac{2}{\pi}\int_{-1}^{1} \frac{d}{dy}(T_i(y))\frac{T_k(y)}{\sqrt{1-y^2}}dy \tag{4.13}$$

and

$$S_k(x,\mu,\theta) = \frac{2}{\pi}\int_{-1}^{1} S(x,y,\mu,\theta)\frac{T_k(y)}{\sqrt{1-y^2}}dy. \tag{4.14}$$

Note that the solutions to the one-dimensional problems given through the equation (4.7)-(4.14) define the components $\Psi_k(x,\mu,\theta)$, for $k=1,...,I$, in this decreasing order to avoid the coupling of the equations. Once this is done, the angular flux given by (4.4) is completely determined. Here we have used the convention $\sum_{i=I+1}^{I}\cdots = 0$. Hence the starting $G_I(x,\mu,\theta) \equiv S_I(x,\mu,\theta)$. Note also that although the solution, developed in here, rely on specific boundary conditions the procedure is quite general in the sense that the expression for the first component, $\Psi_0(x,\mu,\theta)$, keeps the information from the boundary conditions in the y variable, while the other components are derived based on the boundary conditions in x.

Now consider the corresponding discrete ordinates equation [3]

$$\mu_m \frac{\partial^\beta \Psi_\alpha}{\partial^\beta x}(x,\mu_m,\phi_m) + \sigma_t \Psi_\alpha(x,\mu_m,\phi_m) =$$

$$\sum_{n=1}^{M} \omega_n \Psi_\alpha(x,\mu_m,\phi_m) + G_k(x;\mu_m,\phi_m) \tag{4.15}$$

Theorem 4.2. *Let* $0<\beta\leq 1$, *Consider the fractional integro − differential equation* (4.15), *then the function* $\Psi_\alpha(x,\mu_m,\phi_m)$ *satisfies the following first − order fractional linear differential equation for the spatial component* χ_k

$$\mu_m \frac{\partial^\beta \chi_m}{\partial x^\beta} + \left[\sigma_t - \sum_{m=0}^{M}\sqrt{1-\mu_m^2}\left(\sum_{i=\alpha+1}^{I} A_i^\alpha \cos\phi_m - \sum_{j=\alpha+1}^{J} B_j^\alpha \sin\phi_m\right)\right]\chi_m$$
$$= \frac{\pi}{2}\int_{-1}^{1} S_\alpha(x,\mu_m,\phi_m)T_l(\mu_m)d\mu_m.$$

Here χ_k *are the coefficients of the expansion of the* $\Psi_\alpha(x,\mu_m,\phi_m)$.

Proof. We expand $\Psi_\alpha(x,\mu_m,\phi_m)$ in a truncated series of Chebyshev polynomials i.e.

$$\Psi_\alpha(x,\mu_m,\phi_m) = \sum_{k=0}^{M} \frac{\chi_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}} \tag{4.16}$$

bringing the equation (4.16) in equation (4.15) to get

$$\mu_m \frac{\partial^\beta}{\partial x^\beta}\left[\sum_{k=0}^{M} \frac{\chi_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right] + \sigma_t\left[\sum_{k=0}^{M} \frac{\chi_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right] =$$

$$\sum_{n=1}^{M}\omega_n\left[\sum_{k=0}^{M}\frac{\chi_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right]+G_{\alpha}(x;\mu_m,\eta_m) \tag{4.17}$$

with

$$G_\alpha(x;\mu_m,\eta_m)=S_\alpha(x,\mu,\phi)-\sqrt{1-\mu^2}$$

$$\times\left[\cos\phi\sum_{\alpha=i+1}^{I}A_i^\alpha\sum_{k=0}^{M}\frac{\chi_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}+\sin\phi\sum_{\alpha=j+1}^{J}B_j^\alpha\sum_{k=0}^{M}\frac{\chi_k(x,\phi_m)T_k(\mu_m)}{\sqrt{1-\mu_m^2}}\right], \tag{4.18}$$

multiply the equation (4.17) par $T_l(\mu_m)$ and integrate over $\mu_m\in[-1,1]$ we find

$$\mu_m\frac{\partial^\beta}{\partial x^\beta}\sum_{k=0}^{M}\chi_k(x,\phi_m)\int_{-1}^{1}\frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}d\mu_m+\sigma_t\sum_{k=0}^{M}\chi_k(x,\phi_m)\int_{-1}^{1}\frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}d\mu_m=$$

$$\sum_{n=0}^{M}\omega_n\sum_{k=0}^{M}\chi_k(x,\phi_m)\int_{-1}^{1}\frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}d\mu_m+\int_{-1}^{1}G_\alpha(x;\mu_m,\eta_m)T_l(\mu_m)d\mu_m \tag{4.19}$$

with

$$\int_{-1}^{1}G_\alpha(x;\mu_m,\eta_m)T_l(\mu_m)d\mu_m=\int_{-1}^{1}S_\alpha(x,\mu_m,\phi_m)T_l(\mu_m)d\mu_m-\cos\phi_m\sqrt{1-\mu_m^2}\sum_{i=\alpha+1}^{I}A_i^\alpha$$

$$\times\sum_{k=0}^{M}C_k(x,\phi_m)\int_{-1}^{1}\frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}\,d\mu_m+\sin\phi_m\sqrt{1-\mu_m^2}\sum_{j=\alpha+1}^{J}B_j^\alpha$$

$$\times\sum_{k=0}^{M}\chi_k(x,\phi_m)\int_{-1}^{1}\frac{T_k(\mu_m)T_l(\mu_m)}{\sqrt{1-\mu_m^2}}\,d\mu_m \tag{4.20}$$

where

$$A_i^\alpha=\frac{2}{\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{d}{dy}(T_\alpha(y))\frac{T_i(y)}{\sqrt{1-y^2}}T_l(\mu_m)dyd\mu_m \tag{4.21}$$

$$B_j^\alpha=\frac{2}{\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{d}{dz}(T_\alpha(z))\frac{T_j(z)}{\sqrt{1-z^2}}T_l(\mu_m)dzd\mu_m \tag{4.22}$$

by using the properties of Chebyshev polynomials to equation (4.20) to get

$$\int_{-1}^{1}G_\alpha(x;\mu_m,\eta_m)T_l(\mu_m)d\mu_m=$$

$$\int_{-1}^{1}S_\alpha(x,\mu_m,\phi_m)T_l(\mu_m)d\mu_m-\left[\frac{\pi}{2}\sqrt{1-\mu_m^2}\chi_k(x,\phi_m)\right]$$

$$\times \left[\sum_{i=\alpha+1}^{I} A_i^\alpha \cos\phi_m + \sum_{j=\alpha+1}^{J} B_j^\alpha \sin\phi_m \right] \tag{4.23}$$

then the equation (4.15) becomes

$$\mu_m \frac{\partial^\beta \chi_m}{\partial x^\beta} + \left[\sigma_t - \sum_{m=0}^{M} \sqrt{1-\mu_m^2} \left(\sum_{i=\alpha+1}^{I} A_i^\alpha \cos\phi_m - \sum_{j=\alpha+1}^{J} B_j^\alpha \sin\phi_m \right) \right] \chi_m$$

$$= \frac{\pi}{2} \int_{-1}^{1} S_\alpha(x, \mu_m, \phi_m) T_l(\mu_m) d\mu_m. \tag{4.24}$$

after written in vector and matrix notation and regrouping the coefficients χ_m together in equation (4.24), we can derive the following differential equation

$$\frac{\partial^\beta \chi_m}{\partial x^\beta} + DC_m = E_m \tag{4.25}$$

where $D_m = \frac{1}{\mu_m} B_m$ and $E_m = \frac{1}{\mu_m} A_m$ with

$$A_m := \frac{\pi}{2} \int_{-1}^{1} S_\alpha(x, \mu_m, \phi_m) T_l(\mu_m) d\mu_m \tag{4.26}$$

$$B_m := \left[\sigma_t - \sum_{m=0}^{M} \sqrt{1-\mu_m^2} \left(\sum_{i=\alpha+1}^{I} A_i^\alpha \cos\phi_m - \sum_{j=\alpha+1}^{J} B_j^\alpha \sin\phi_m \right) \right] \tag{4.27}$$

the solution of differential equation for the vector χ_m is thus constructed as follows [1]

$$\chi_m(x) = e_\beta^{-Dx} \chi_m(0) + \int_0^x e_\beta^{-(x-\xi)D} E_m(\xi) d\xi \tag{4.28}$$

where

$$e_\beta^{Dx} = x^{\beta-1} \sum_{k=0}^{\infty} D^k \frac{x^{k\beta}}{\Gamma\left[(k+1)\beta\right]} \tag{4.29}$$

equation (4.28) depend on vector $\chi_m(0)$. Having established an analytical formulation for the exponential appearing in equation (4.29), the unknown components of vector $\chi_m(0)$ for the boundary problem (4.1) can be readily obtained applying the boundary conditions (4.2), (4.3) and (4.4).

5 Study of the spectral approximation

Now we expand

$$\Psi_{\alpha,N}(x, \phi_m) = \sum_{m=0}^{(N)} \chi_m^{(N)} \cos(m\phi_m) \tag{5.1}$$

where $\chi_m^{(N)}$ is the approximation to the coefficient χ_m by the consideration of the truncated series $\Psi_{\alpha,N}$. From spectral analysis, we know that when a function is infinitely smooth and all its derivatives exist, then the coefficients appearing in its sine or cosine series go to zero faster than $1/n$. Moreover, if the function and all its derivatives are periodic, then the decay is faster than any power of $1/n$. However, as indicated by Canuto. (1988) [2], in practice this decay cannot be observed before enough coefficients that represent the essential structures of the function are considered.

In the calculation, one can test the convergence of the cosine truncated series defined in equation (5.1) by evaluating

$$\sup_k \left[\frac{\mid \Psi_{N+1}(k) - \Psi_N(k) \mid}{\Psi_N(k)} \right] \leq \epsilon \tag{5.2}$$

where ϵ is the required precision. In general, the few first coefficients of the series are enough to generate the angular flux.

If N is the chosen value, we can write

$$\chi_m^{(N)} = 0 \qquad \text{for all} \qquad n > N, \tag{5.3}$$

Combining therefore equations (5.3) and (4.28) we shall now describe the necessary algorithm to obtain all the cosine coefficients $\chi_m^{(N)}$

Step 0: $N = 0$; for $n = N = 0$

$$\chi_0^{(0)}(x) = e_\beta^{-Dx} \chi_0^{(0)}(0) - \int_0^x e_\beta^{-D(x-s)} A_0(x) dx, \tag{5.4}$$

with

$$A_0 := \pi \int_{-1}^{1} S_\alpha(x, \mu_0, \phi_0) T_l(\mu_0) d\mu_0 \tag{5.5}$$

which is well known, and thus $\chi_0^{(0)}(x)$ is completely determined. To finish the step, we apply equation (5.1) to obtain the first approximation to the angular flux, i.e., Ψ_0.

Step 1: $N = 1$; for $n = 0$,

$$\chi_0^{(1)}(x) = e_\beta^{-Dx} C_0^{(1)}(0) - \int_0^x e^{-D(x-s)} A_1(x) dx, \tag{5.6}$$

with

$$A_1 := \frac{\pi}{2} \int_{-1}^{1} S_\alpha(x, \mu_1, \phi_1) T_l(\mu_1) d\mu_1 \tag{5.7}$$

Step 2: for $n = 1$

$$\chi_1^{(1)}(x) = e_\beta^{-Dx} \chi_1^{(1)}(0) - \int_0^x e_\beta^{D(x-s)} A_1(x) dx, \tag{5.8}$$

with

$$A_1 := \frac{\pi}{2} \int_{-1}^{1} S_\alpha(x, \mu_1, \phi_1) T_l(\mu_1) d\mu_1 \tag{5.9}$$

Bringing the approximated solution for $\chi_0^{(0)}$ obtained at step 0 inside equation (5.8) and iterating with equation (5.4), we obtain immediately the approximated coefficients $\chi_0^{(1)}$and $\chi_1^{(1)}$. To finish the step, we evaluate through equation (5.1) the new approximation Ψ_1 and perform the precision condition defined in equation (5.2). If equation is verified, the calculation is stopped; if not, we go to step 2 and to likewise until the convergence condition in equation (5.2) is fulfilled.

6 Conclusions and future work

In this paper, we have shown that we can reduce a fractional transport equation. It is interesting to mention that we can extend this procedure for higher spatial dimensions. These will be our future investigations.

References

[1] B. Bonilla, M. Rivero, and J. J. Trujillo, *On systems of linear fractional differential equations with constant coefficients*, Applied Mathematics and Computation 187, 1 (2007), 68-87.

[2] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Mechanics*, New York: Springer, 1988.

[3] S. Chandrasekhar, *Radiative Transfer*, Dover, New York, 1960.

[4] A. Kadem, *Solving the one-dimensional neutron transport equation using Chebyshev polynomials and the Sumudu transform*, Anal. Univ. Oradea, fasc. Math. 12 (2005), 153-171.

[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, 2006.

[6] Y. Luchko, and R. Gorenflo, *An operational method for solving fractional differential equation with the Caputo derivatives*, Acta Math. Vietnamica 24, 2 (1999), 207-233.

[7] K. Oldham, and J. K. Spanier, *The Fractional Calculus*, Academic Press, New York. 1974.

[8] F. W. J. Olver, *Asymptotic and Special Functions*, Academic Press, New York, 1974.

[9] G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[10] Z. Trzaska, *An efficient algorithm for partial expansion of the linear matrix pencil inverse*, J. of the Franklin Institute 324 (1987), 465-477.

[11] G. K. Watugala, *Sumudu transform - a new integral transform to solve differential equations and control engineering problems*, Math. Engrg. Indust. 6, 4 (1988), 319-329.

[12] J. Zabadal, M. T. Vilhena, and P. Livotto, *Simulation of chemical reaction using fractional derivatives*, II Nuovo Cimento 116-B, 3 (2001a).

Author's address:

Abdelouahab Kadem
L.M.F.N Mathematics Department,
University of Setif, Algeria.
E-mail: abdelouahabk@yahoo.fr