Berwald-Moor metrics and structural stability of conformally-deformed geodesic SODE

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Abstract. The robustness and the fragility of a second-order SODE is studied by means of the five KCC invariants, which provide information on the Jacobi stability ([1, 8, 53, 14]). The paper presents a brief overview on Finslerian $m-$root metrics, in particular, of Berwald-Moor type, and applies the developed theory to a Berwald-Moor type conformally deformed relativistic model, where the five KCC invariants of the Finslerian framework are computed by means of the MAPLE 12 symbolic software.

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1 Brief account on the KCC theory

The KCC (also called structual, or Jacobi) stability, adds a complementary degree of accuracy to the classical linear analysis, by studying the robustness and fragility of a second-order SODE ([1]-[8],[53]-[54],[10]-[15]). By robustness are assumed both the relative insensitivity to alteration of the internal parameters and the ability to adapt to changes in environment. From the mathematical point of view, the differential geometrical theory of variational equations - which studies the deviation of nearby trajectories, allows us to estimate the admissible perturbation around the steady states of the SODE. In the real world applications is of interest to identify the "robust arrest" regions - i.e., the regions where one has both Lyapunov and Jacobi stability. This can be achieved by means of the KCC theory.

The roots of the KCC theory reside in the works of D.D. Kosambi [28], E. Cartan [21] and S.S.Chern [22, 23], and the abbreviation KCC (Kosambi-Cartan-Chern) emerges from the names of the three initiators of the framework.

It is noteworthy to stress that the first attempts to settle and to develop systematically the KCC theory are due to P.L. Antonelli and I. Bucătaru ([1],[4] and [5]). In [4] it was proved that all the five KCC invariants of the SODE can be expressed in the Finslerian geometric framework, and using the dynamical covariant derivative and the covariant derivative induced by the Berwald connection are determined two

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invariant equations for the variational equations of a SODE and for the symmetries of the associated semispray.

The most significant applications of the KCC theory have been developed for second-order autonomous systems (2.1) in which g^i (and hence the nonlinear semispray components $G^i(x,y)$) are quadratic affine forms in terms of y. For several such systems which provide Lotka-Volterra models from biology, the Jacobi stability has been investigated ([3, 5, 6, 3]).

The KCC theory has been applied in population genetics, engineering, ecology ([7, 44]), in plasma physics ([24]) and in Belousov-Zhabotinskii reaction model in chemistry ([65]). Recent advances have been obtained in the Riemannian KCC framework, in stating the exact relations which exist between the stability equation for the solutions of a mechanical system and the geodesic deviation equation of the associated geodesic problem in the Jacobi metric constructed via the Maupertuis-Jacobi Principle, and concluding that the dynamical and geometrical approaches to the stability/instability problem are not equivalent ([26]). An application of the KCC theory was developed in [64], where the the Jacobi stability of a Rikitake system associated to a two-disk dynamo system was studied and where it was shown that the chaotic behavior of the magnetic field can be investigated using the five KCC invariants.

Numerous applications of the KCC theory to processes from biology which are described by second order SODE ([2, 29]) and by second-order extensions of dynamical systems (e.g., [12, 13, 15, 43, 45]), have been developed. As well, intensive research work has been provided on the Jacobi stability of gyroscopic-type second-order extensions of dynamical systems ([61]) which model biological processes, and of their linearizations ([42, 62, 63]).

2 The KCC structural invariants

We provide hereafter the main basics of KCC theory ([1], [53], [4]). Let $x = (x^1, ..., x^n)$ be the components of a curve in $\mathbb{R}^n$, and consider its velocity $\dot{x} = dx/dt$. We further assume that $(x, y = \dot{x}, t)$ belong to an open connected subset $\mathbb{D} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^1 \equiv \mathbb{R}^{2n+1}$, and will denote the coordinates in this region using the components of the $(2n+1)-$tuple (x, y, t). In the following sections, position x will lay in a local chart of an $n-$dimensional smooth real manifold M, endowed with Finslerian structure F.

Consider a second order system of differential equations in normalized form

$$\frac{d^2x^i}{dt^2} + g^i(x, \dot{x}, t) = 0, \quad i \in \overline{1, n}, \tag{2.1}$$

where $g^i(x, \dot{x}, t), i \in \overline{1, n}$ are smooth functions defined in a neighborhood of some initial conditions $(x_0, \dot{x}_0, t_0) \in \mathbb{D}$. In order to find the KCC differential invariants of the system (2.1) under the non-singular coordinate transformations of the form

$$\overline{x}^i = f^i(x^1, \dots, x^n), i \in \overline{1, n}, \quad \bar{t} = t, \tag{2.2}$$

we define *the KCC covariant differential of a contravariant vector field* ξ^i on $\mathbb{D}$ via

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + \frac{1}{2} g^i_{;r} \xi^r, \tag{2.3}$$

where ";" indicates partial differentiation with respect to $\dot{x}$, and the Einstein summation convention is implicitly assumed. Using (2.3) for $\xi^r = \dot{x}^r$, the initial system (2.1) becomes

$$\frac{D\dot{x}^i}{dt} = \varepsilon^i, \quad \text{with} \quad \varepsilon^i = \frac{1}{2}\, g^i_{;r}\, \dot{x}^r - g^i. \tag{2.4}$$

The contravariant vector field ε defined on $\mathbb{D}$ is called *the first KCC invariant*, and plays the rôle of an external force ([1]). We note that ε identically vanishes iff the functions $g^i = g^i(x, \dot{x}, t)$ are 2-homogeneous in $\dot{x}$. Hence, in geometric terms $\varepsilon \equiv 0$ is a necessary and sufficient condition for the semispray defined by (g^i), to be a spray. It is obvious that for the *geodesic* spray of a Riemannian or Finsler manifold, this relation is fulfilled.

Further, if one varies the paths $x^i(t)$ of (2.1) with respect to x as $\bar{x}^i(t) = x^i(t) + \xi^i(t)\eta$, with $0 < |\eta| << 1$, then the following variational equations are derived

$$\frac{d^2\xi^i}{dt^2} + g^i_{;r}\frac{d\xi^r}{dt} + g^i_{,r}\xi^r = 0, \tag{2.5}$$

where "," indicates partial differentiation with respect to x. We can write (2.5) in terms of the KCC covariant differential (2.3), in the covariant form

$$\frac{D^2\xi^i}{dt^2} = P^i_r\xi^r, \tag{2.6}$$

where the right hand side (1,1)-tensor

$$P^i_j = -g^i_{,j} - \frac{1}{2}g^r g^i_{;r;j} + \frac{1}{2}\dot{x}^r g^i_{,r;j} + \frac{1}{4}g^i_{;r}g^r_{;j} + \frac{1}{2}\frac{\partial g^i_{;j}}{\partial t} \tag{2.7}$$

is called *the second KCC invariant* of the system (2.1), or *deviation curvature tensor*. Its eigenstructure provides an alternative to the Floquet Theory ([53]), with the eigenvalues of P^i_j replacing the characteristic multipliers (also called Floquet exponents, [49], [8]).

The usage of the term *Jacobi stability* within the KCC theory is justified by the fact that, when the emerging system (2.1) represents the SODE of geodesic equations in Finsler or Riemannian frameworks ([17, 37, 31, 50]), then (2.6) is the Jacobi field equation. Specifically, the Jacobi equation (2.6) of the Finsler manifold (M, F) can be written in the scalar form ([17]):

$$\frac{d^2v}{ds^2} + K \cdot v = 0, \tag{2.8}$$

where $\xi^i = v(s)\eta^i$ is a Jacobi field along the geodesic $x^i(s)$, η^i is the unit normal vector field, and K is the flag curvature of (M, F). The sign of K influences the geodesic rays: if $K > 0$, then the geodesics bunch together (are Jacobi stable), and if $K < 0$, then they disperse (are Jacobi unstable).

Hence negative/positive flag curvature is equivalent to positive/negative eigenvalues of P^i_j. In this respect, it is known the following result ([6], [1]):

Theorem 2.1. *The integral curves of the second-order SODE (2.1) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P^j_i are strictly negative everywhere, and Jacobi unstable, otherwise.*

The notion of Jacobi stability presented above can be extended to the general case of the SODE (2.1) using this Theorem as definition for the Jacobi stability of the integral curves of a rheonomic second-order SODE.

The third, fourth and fifth KCC invariants of the system (2.1) are respectively

$$R^i_{jk} = \frac{1}{3}(P^i_{j;k} - P^i_{k;j}), \quad B^i_{jkl} = R^i_{jk;l}, \quad D^i_{jkl} = g^i_{;j;k;l}. \tag{2.9}$$

A basic result of the KCC theory which points out the rôle of the five invariants ε^i, P^i_j, R^i_{jk}, B^i_{jkl} and D^i_{jkl} respectively defined by (2.4), (2.7), (2.9), is the following ([1]):

Theorem 2.2. *Two SODEs of the form (2.1) on $\mathbb{D} \subset \mathbb{R}^{2n+1}$ can be locally transformed relative to (2.2), one into another, if and only if their five invariants are equivalent tensors. In particular, there exists a local system of coordinates $(\bar{x})$, for which $g^i(x, \dot{x}, t) = 0$, if and only if all the five KCC tensors vanish.*

If there exist no coordinate change such that the coefficients of the new second order SODE-semispray do all vanish, according to ([1]), this implies that the integral curves of the second-order extended system can never be straight lines, whatever coordinate system one might choose.

The study of Jacobi stability is complementary to the study of linear stability and is based on the study of Lyapunov stability of whole trajectories in a region, and hence the perturbation yields trajectories close to the reference trajectory ([42, 62, 63]). Similarly, in the case of Lyapunov stability, the perturbations of a stable equilibrium point lead to trajectories which will be dumped out, provided that these are small enough so as not to escape from a basin of attraction (see [6, 53]).

3 The KCC Finslerian framework

The KCC theory of a system of second order ordinary differential equations uses five geometric invariants that determine, up to a change of coordinates, the solutions of the system. The vector field (semispray) associated to the SODE determines a nonlinear connection N and a Berwald connection Γ, both considered on TM. A particular case when the semispray is homogeneous in y, appears e.g. for the canonic Cartan nonlinear connection on a Finsler manifold ([4]). We shall briefly describe below the geometry of the second-order SODE (2.1) in the Finslerian framework.

Let (M, F) be an $n-$dimensional Finsler space with fundamental function $F : TM \to \mathbb{R}$ ([37, 18]) satisfying the following conditions:

i) F is of class C^0 on TM and C^∞ on the the slit tangent space $\widetilde{TM} = TM \backslash \{0\}$;

ii) F is positive homogeneous in $y \in T_xM$: $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$;

iii) [1] The fundamental metric tensor field $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$ where $L = F^2$, is non-degenerate and has constant signature, where we have denoted $\dot{\partial}_j = \partial/\partial y^j$.

We shall further assume that the raising/lowering of indices is performed by means of the fundamental metric tensor field. The *equations of motion in a Finsler space* entailed by the action of a force $\Lambda = (\Lambda_i)$ provided by a Finsler 1-form, are given by a second-order SODE

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = \frac{1}{2}\Lambda_i, \quad i = \overline{1,n}, \tag{3.1}$$

where $\Lambda_i \frac{\partial}{\partial y^i}$ is the external force ([38, 20]), which vanishes in the case of geodesic equations. We note that these equations can be written as a second-order SODE of type (2.1):

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = 0, \quad i \in \overline{1,n}, \tag{3.2}$$

where the functions $\{G^i\}$ determine a semispray

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, \tag{3.3}$$

which provide the coefficients of a nonlinear connection N ([56]), via

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{3.4}$$

It is known that a sufficiency condition for the trajectories of the dynamical system given locally on TM by the semispray S,

$$\begin{cases} \dot{x}^i = y^i \\ \dot{y}^i = -2G^i(x,y) \end{cases} \tag{3.5}$$

for not behaving chaotically, is that hte SODE admits a constant of motion. In this respect the following result ([55]), which is used in KCC-related applications (e.g., in [64]), holds true:

Theorem 3.1. *Assume that there exists a vector field X on M such that $[X, S] = f\,S$ for some function f on M. Then*

$$S(f + div_\Omega X) = -f\, div_\Omega S + X(div_\Omega S),$$

where S is the semispray (3.3), and div_Ω is the divergence operator w.r.t. the volume form Ω on M.

As a consequence of this theorem, for $f = 0$ and $\mathrm{div}_\Omega S$ constant, the system (3.5) admits the *constant of motion* $\mathrm{div}_\Omega X$.

[1] this property does not assume positive definiteness; the Finsler spaces of non-positive signature, like several of investigated spaces of Berwald-Moor type might be called *pseudo-Finsler* ([19]).

The coefficients (3.4) define the KCC covariant differential of a vector field $\xi^i(t)$, as

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N^i_j \xi^j, \tag{3.6}$$

which applied to the Liouville vector field $y^i \frac{\partial}{\partial y^i}$ - considered along the integral curves of (3.5), leads to the alias of the system (2.4), $\frac{Dy^i}{dt} = \varepsilon^i$, where $\varepsilon^i = N^i_j y^j - 2G^i$ coincides with the first KCC invariant (3.2), canonically related to the generalized force Λ ([4]). We note that in the absence of forces ($\Lambda \equiv 0$), one gets the equations of geodesics of the Finsler space. In this case, applying the variation process described in the previous section to the integral paths of (3.5), one obtains the equations

$$\frac{d^2\xi^i}{dt^2} + 2N^i_j \frac{d\xi^j}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0,$$

which can be written in terms of the KCC covariant derivative (3.6) as

$$\frac{D^2\xi^i}{dt^2} = P^i_j \xi^j. \tag{3.7}$$

The right term exhibits the components of the second KCC tensor,

$$P^i_j = 2\frac{\partial G^i}{\partial x^j} + 2G^s \Gamma^i_{js} - \frac{\partial N^i_j}{\partial x^s} y^s - N^i_s N^s_j - \frac{\partial N^i_j}{\partial t},$$

where $\Gamma^i_{jk} = \dot{\partial}_k N^i_j$ are the components of the Berwald connection. The last three KCC invariants are provided by

$$R^i_{jk} = \frac{1}{3}\dot{\partial}_{[k} P^i_{j]}, \quad B^i_{jkl} = \dot{\partial}_l P^i_{jk}, \quad D^i_{jkl} = \dot{\partial}_l \Gamma^i_{jk},$$

where we have denoted $\tau_{[i\ldots j]} = \tau_{i\ldots j} - \tau_{j\ldots i}$ and $\dot{\partial}_k = \partial/\partial y^k$.

The third invariant (R^i_{jk}) is skew-symmetric in the lower indices, and is regarded as $hh - v$ torsion tensor (*the curvature of the nonlinear connection*, [56, 37]), the fourth invariant (B^i_{jkl}) is the *Berwald horizontal Finslerian curvature* tensor, while (D^i_{jkl}) is *the Douglas tensor*. These represent fundamental geometric objects in Finsler Geometry ([31, 50, 37]).

4 Locally Minkowski Finsler m-root metrics

As a natural extension of the study of the cubic Locally Minkowski Finsler metric ([33]), numerous results have been derived for the genera case, when the Finsler metric of the space (M, F) has the form

$$F(x, y) = \sqrt[n]{a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}}, \tag{4.1}$$

where $a_{i_1 i_2 \cdots i_m}(x)$, depending on the position alone, is symmetric in all its indices $i_1, i_2, \ldots, i_m$ and $m \geq 3$. We shall call the Finsler space with the metric (4.1) an *m-root Finsler space*. In case of $m = 3$, it is called a *cubic* Finsler space [33].

4.1 S_3-likeness

We shall further consider the m-root metric (4.1). We use the following notations

$$\begin{cases} a_i = a_{ii_2\dots i_m}(x)y^{i_2}\dots y^{i_m}/F^{m-1}, \\ a_{ij} = a_{iji_3i_4\dots i_m}(x)y^{i_3}y^{i_4}\dots y^{i_m}/F^{m-2}, \\ a_{ijk} = a_{ijki_4i_5\dots i_m}(x)y^{i_4}y^{i_5}\dots y^{i_m}/F^{m-3}. \end{cases} \tag{4.2}$$

The *normalized supporting element* $l_i = \dot{\partial}_i F$, where $\dot{\partial}_i = \partial/\partial y^i$, the *fundamental tensor* $g_{ij} = \dot{\partial}_i\dot{\partial}_j F^2/2$ and *the angular metric tensor* $h_{ij} = F\dot{\partial}_i\dot{\partial}_j F$ for the $m-$root metric have the form, respectively,

$$\begin{cases} l_i = a_i \\ g_{ij} = (m-1)a_{ij} - (m-2)a_i a_j \\ h_{ij} = (m-1)(a_{ij} - a_i a_j). \end{cases} \tag{4.3}$$

We suppose that the determinant of the tensor a_{ij} does not vanish, i.e., a_{ij} is *regular* [33]. If we denote by a^{ij} the dual components ($a^{ij}a_{jk} = \delta^i_k$), then the reciprocal components g^{ij} of g_{ij} are given by

$$g^{ij} = \frac{1}{m-1}[a^{ij} + (m-2)a^i a^j], \tag{4.4}$$

where we put $a^i = a^{ij}a_j$. In the above we used the relation $a^2 = a_i a^i = 1$ ([33]). It follows from (4.3) that the tensor $C_{ijk} = \dot{\partial}_k g_{ij}/2$ has the particular form

$$C_{ijk} = \frac{(m-1)(m-2)}{2F}(a_{ijk} - a_{ij}a_k - a_{ik}a_j - a_{kj}a_i + 2a_i a_j a_k), \tag{4.5}$$

and, using (4.4) and (4.5), the components $C_j{}^i{}_k = C_{jrk}g^{ir}$ are in the $m-$root case

$$C_j{}^i{}_k = \frac{m-2}{2F}[a_j{}^i{}_k - (\delta^i_j a_k + \delta^i_k a_j) + a^i(2a_j a_k - a_{jk})], \tag{4.6}$$

where $a_j{}^i{}_k = a_{jrk}a^{ir}$. Then the torsion vector $C_i = C_i{}^r{}_r$ is given by

$$C_i = \frac{m-2}{2F}(a_i{}^r{}_r - na_i).$$

Remark. In case of the Finsler space with the Berwald-Moor metric $L = (y^1y^2\dots y^n)^{1/n}$, which is a special $m-$root metric, the torsion vector C_i vanishes ([36]).

From (4.5), (4.6), the v-curvature tensor $S_{hijk} = C_h{}^r{}_k C_{rij} - C_h{}^r{}_j C_{rik}$ of the Cartan connection $C\Gamma$ ([32, (17.20)]) has the form

$$S_{hijk} = \frac{1}{4F^2}(m-1)(m-2)^2\mathcal{A}_{(jk)}\{a_h{}^r{}_k a_{rij} - a_{ij}(a_{hk} - a_h a_k) + a_i a_j a_{hk}\}, \tag{4.7}$$

where $\mathcal{A}_{(jk)}$ means interchange of indices j, k and subtraction. By virtue of (4.3), (4.7) is rewritten in the form

$$S_{hijk} = \frac{1}{4F^2}(m-2)^2[(h_{hj}h_{ik} - h_{hk}h_{ij})/(m-1) + (m-1)H_{hijk}], \tag{4.8}$$

where we put $H_{hijk} = a_{h\ k}^{\ r} a_{rij} - a_{h\ j}^{\ r} a_{rik}$. A Finsler space of dimension $n \geq 4$ is called *S3-like* [32], if there exists such a scalar S that the v-curvature tensor S_{hijk} is written in the form

$$S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij})/F^2.$$

4.2 Particular cases of m-root metric

a) The $m-$root Finsler metric (4.1), for which $a_{i_1 i_2 \ldots i_m}(x)$ is zero, has the explicit form

$$F(y) = \sqrt[m]{(y^1)^m + (y^2)^m + \ldots + (y^n)^m}. \tag{4.9}$$

We note that for $m = 4$ this metric represents the historic primary Finsler fundamental function, considered by B. Riemann in his "Habilitation address" ([58]). For this case the following result holds true ([46, 9]):

Theorem 4.1. *An $n-$dimensional Finsler space with the metric (4.9) has the following properties:*
i) is $S3-$like, provided that $y^1 y^2 \ldots y^n \neq 0$, and the $v-$curvature tensor S_{hijk} has the form

$$S_{hijk} = \frac{(m-2)^2}{4F^2(m-1)}(h_{hj}h_{ik} - h_{hk}h_{ij}). \tag{4.10}$$

ii) the indicatrix is of constant curvature $m^2/4(m-1)$.

b) Another example is the metric considered by G.S. Asanov ([9]), which represents an extension of (4.9),

$$F(y) = \sqrt[m]{\varepsilon^1 (y^1)^m + \varepsilon^2 (y^2)^m + \ldots + \varepsilon^n (y^n)^m},$$

where $\varepsilon^k \in \mathbb{R}$. This metricis $S3-$like, provided that $m \geq 3$ and $\varepsilon^i \neq 0, \forall i \in \overline{1,n}$, and the $v-$curvature tensor S_{hijk} has the form (4.10).

c) For $m = n$, the Berwald-Moor relativistic metric ([36, 47, 48]) is the pseudo-Finslerian structure provided by the Shimada metric (4.1) in which

$$a_{i_1 i_2 \ldots i_m}(x) = \begin{cases} 1/n!, & \{i_1, \ldots, i_m\} = \{1, \ldots, m\} \\ 0, & \text{otherwise.} \end{cases}$$

The fundamental function has the explicit expression

$$F(y) = (y^1 y^2 \ldots y^n)^{1/n}, \tag{4.11}$$

and it is known ([9]) that the $v-$curvature tensor S_{hijk} is

$$S_{hijk} = -(h_{hj}h_{ik} - h_{hk}h_{ij})/F^2 \tag{4.12}$$

and further $A_i = LC_i$ vanishes. It follows that the curvature of the indicatrix I_x vanishes as well, $R_{\alpha\beta\gamma\delta} \equiv 0$, i.e. I_x is flat. Summarizing, the following holds true

Theorem 4.2. *An $n-$dimensional Finsler space with the Berwald-Moor metric is $S3-$like, is $P-$symmetric and has flat indicatrix I_x, for all $x \in M$.*

Moreover, it has been shown ([9]), that the extension

$$F(y) = (y^1)^{\varepsilon^1}(y^2)^{\varepsilon^2} \cdot \dots \cdot (y^n)^{\varepsilon^n}, \quad \varepsilon^1 + \dots + \varepsilon^n = 1,$$

is $S3-$like as well and admits of the form (4.10).

4.3 Decomposability of $m-$root metrics

We say that a Finsler tensor field $v_{ij}(x,y)$ is *$h-$recurrent*, if it satisfies a relation of the form $v_{ij|k} = \gamma_k v_{ij}$, where $\gamma_k(x,y)$ is a Finsler 1-form and $|$ denotes the $h-$covariant derivative. The following results are known ([35])

Theorem 4.3. *Let (M,F) be an $n-$dimensional Finsler space with the metric $F(x,y)$.*
a) *If $n = 2m$ and*

$$F = (\alpha^2_{(1)}\alpha^2_{(2)}...\alpha^2_{(m)})^{1/2m},$$

where $\alpha^2_{(a)} = \alpha_{(a)ij}(x)y^iy^j, \forall a \in \overline{1,m}$ are quadratic forms induced by independent Riemann metrics, then (M,F) is a Berwald space (i.e., $C_{ijk|l} = 0$), iff for all $a \in \overline{1,m}$, $\alpha_{(a)ij}(x)$ are $h-$recurrent and

$$\alpha_{(a)ij|k} = \gamma_{(a)k}(x)\alpha_{(a)ij}, \quad \gamma_{(1)k} + \dots + \gamma_{(m)k} = 0.$$

b) *If $n = 2m+1$ and*

$$F = (\alpha^2_{(1)}\alpha^2_{(2)}...\alpha^2_{(m)}\beta)^{1/2m+1},$$

where $\alpha^2_{(a)}$ are as above and $\beta = \beta_i(x)y^i$ is a 1-form, then (M,F) is a Berwald space, iff all the Riemannian metric tensors $\alpha_{(a)ij}$ and the covariant vector field β_i are $h-$recurrent such that for any $a = \overline{1,m}$, we have

$$\begin{cases} \alpha_{(a)ij|k} = \gamma_{(a)k}(x)\alpha_{(a)ij}, \\ \beta_{i|k} = \gamma_{(m+1)k}(x)\beta_i \end{cases}, \quad \gamma_{(1)k} + \dots + \gamma_{(m+1)k} = 0.$$

c) *If $F = (a^1a^2...a^n)^{1/n}$, where $a^\alpha = a^\alpha_i(x)y^i$ $(\alpha = \overline{1,n})$ are independent 1-forms, then (M,F) is a Berwald space, iff the covariant vector fields a^α_i are $h-$recurrent and*

$$a^\alpha_{i|j} = \gamma^\alpha_j(x)a^\alpha_i, \quad \gamma^1_j + \dots + \gamma^n_j = 0.$$

Remark. One can easily notice that the theorem holds true for the typical Berwald-Moor metrics $F_n(y) = \sqrt[n]{y^1 \cdot \dots \cdot y^n}$ in $\mathbb{R}^n$, with the proof adapted for *pseudo-Riemannian metrics* $\alpha_{(a)ij}(x)$. For $n = 4$ items a) and c), and for $n = 3$ items b) and c) hold true. The horizontal derivatives of the generating 1- and 2-forms all cancel which yield vanishing coefficients γ. This is due to the fact that the metric coefficients are $x-$independent and the extended Christoffel symbols identically vanish, which entails the cancelling both of the Cartan canonic nonlinear connection and of the horizontal Cartan connection coefficients.

4.4 The conformally deformed Berwald-Moor metric

We shall further consider a deformed Berwald-Moor Finsler structure (M, F) with $\dim M = 4$, for which was designed the developed Maple 12 symbolic package which computes the KCC invariants of the geodesic SODE. The basic fundamental Finsler function (called *4-root metric*, *quartic metric* or *4-th order Shimada metric*) is of the form

$$F(x,y) = \sqrt[4]{a_{i_1\dots i_4}(x) y^{i_1} y^{i_2} y^{i_3} y^{i_4}}, \tag{4.13}$$

with the functions $a_{i_1\dots i_4}$ symmetric in all their indices. In our computations, we have considered two special cases of quartic metrics, namely

$$\begin{aligned} F_0(y) = \; & [(y^1+y^2+y^3+y^4)(y^1+y^2-y^3-y^4)\cdot \\ & \cdot (y^1-y^2+y^3-y^4)(y^1-y^2-y^3+y^4)]^{1/4}. \end{aligned} \tag{4.14}$$

and

$$F_0(y) = \sqrt[4]{y^1 \dots y^4}.$$

Starting with F_0, we construct the conformally deformed *Shimada metric*

$$F(x,y) = e^{2\sigma(x)} F_0(y). \tag{4.15}$$

Then, the second-order SODE which describes the geodesics of the space (M, F) is:

$$\frac{d^2 x^i}{dt^2} + \gamma^i_{00} = 0, \quad i = \overline{1,n},$$

where the null index denotes transvection with y^i, and in the SODE, with $y^i = \frac{dx^i}{dt} = \dot{x}^i$. We denote as well with γ^i_{jk} the generalized Christoffel symbols

$$\gamma^i_{jk} = \frac{1}{2} g^{is}(g_{\{js,k\}} - g_{jk,s}) = \overset{\circ}{\gamma}{}^i_{jk} + (\delta^i_{\{k}\sigma_{j\}} - \sigma^i g_{jk}), \tag{4.16}$$

where $\tau_{\{ij\}} = \tau_{ij} + \tau_{ji}$, δ^i_j is the Kronecker symbol, $\sigma^i = g^{is}\sigma_s$, $g_{ij} = 2^{-1}(F^2)_{;ij}$, "," denotes the partial derivative w.r.t. x^i, ";" denotes partial derivative w.r.t. y^i, and $\overset{\circ}{\gamma}{}^i_{jk} \equiv 0$ are components of the flat Christoffel symbols of the Locally Minkowski Finsler metric structure (M, F_0).

Then one easily infers $\overset{\circ}{\gamma}{}^i_{00} = \gamma^i_{00} + 2y^i\sigma_* - \sigma^i F_0^2$ with $\sigma_* = \sigma_s\sigma^s$, and further, the explicit form of the geodesic SODE

$$\ddot{x}^i + \gamma^i_{00} = \sigma^i F_0^2(x, \dot{x}) - 2\dot{x}^i \sigma_*, \quad i = \overline{1,n}, \tag{4.17}$$

which exhibits in the right hand side the force which governs the displacement of the test-particles of (M, F) along the geodesics, deviating the original rectilinear trajectories. In simulations, we have used the linear potential $\sigma = \sum_{i=\overline{1,n}} x^i$, for which the geodesic SODE (4.17) considerably simplifies to

$$\ddot{x}^i + \gamma^i_{00} = F_0^2(x, \dot{x}) \cdot \sum_{s=\overline{1,n}} g^{is} - 2\dot{x}^i \cdot \sum_{s,r=\overline{1,n}} g^{sr}, \quad i = \overline{1,n}. \tag{4.18}$$

Remark. The deformation algebra provided by the pair of generalized connections $\overset{\circ}{\gamma}{}^{i}_{jk} \equiv 0$ and γ^i_{jk} given by (4.16) is obviously commutative, and is associative iff

$$\gamma^i_{[ls}\gamma^s_{r]j} = 0, \forall\, i, j, l, r \in \overline{1, n},$$

where $\tau_{[ij]} = \tau_{ij} - \tau_{ji}$, which relies to

$$\delta^i_{[l}\sigma_{r]}\sigma_j - \sigma_* \delta^i_{[l}g_{r]j} + \sigma^i \sigma_{[l}g_{r]j} = 0,$$

which for i/l transvection leads to $g_{rj} = \sigma_r\sigma_j/\sigma_*$, possible only for $n = 1$.

As well, direct computation shows that the Finslerian vector field $X^i = \sigma^i \frac{\partial}{\partial y^i}$ is not a 0-divisor in the non-associative conformal deformation algebra (the parallelism along the field lines of the gradient lines of σ is always non-Euclidean).

5 KCC Maple symbolic software

The symbolic software Maple 12 provides a convenient environment for the calculation of Finslerian relativistic geometric objects, as extension to the *GRTensorII* package ([66]), which provides the tensors involved in the Einstein equations of the Riemannian framework. The Finslerian version has been achieved by S.F. Rutz (the computer algebra package FINSLER, [1, 51, 52]). The proposed software manages to explicitly determine the five KCC invariants of the SODE (4.18) for the Finslerian framework. The Maple 12 source code is included in the appendix.

6 Conclusions

The present work includes a brief presentation of the KCC theory, an outlook on the main properties of m-root Finsler structures - mainly based on the works of M. Matsumoto, H. Shimada, S. Numata and K.Okubo, and presents the equation of motion in the conformally deformed 4-dimensional Berwald-Moor Finsler framework; in the appendix are included the original Maple procedures developed for determining the first five KCC invariants for conformally deformed $m-$root metrics.

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Appendix.

Maple package for the five KCC invariants (4D Finslerian framework)

```
Preliminaries
> restart: with(linalg): with(LinearAlgebra): with(tensor):
Build y_t, y[i], i=1..4
> y:=array(1..4); y[1]:=y1: y[2]:=y2: y[3]:=y3: y[4]:=y4:
> y_t:=create([1], eval(y)); type(y_t,tensor_type);
> y:=get_compts(y_t);
Build x_t,  x[i]
> x:=array(sparse,1..4); x[1]:=x1: x[2]:=x2: x[3]:=x3:
> x[4]:=x4:    x_t:=create([1], eval(x)); x:=get_compts(x_t);
Build F0 & F (Finsler metrics)
> F0:=root((y[1]+y[2]+y[3]+y[4])*(y[1]+y[2]-y[3]-y[4])*
       (y[1]-y[2]+y[3]-y[4])*(y[1]-y[2]-y[3]+y[4]),4);
>#F_0=root((y[1]*y[2]*y[3]*y[4]),4);
> apply(a,(x[1],x[2],x[3],x[4]));
> F:=apply(a,(x[1],x[2],x[3],x[4]))*F0;
Build g (g[i,j])
> g_compts:=array(symmetric,1..4,1..4);
> for i from 1 to 4 do for j from 1 to 4 do
>   g_compts[i,j]:=simplify((diff(F^2,y[i],y[j]))/2);
>   end do; end do;
> g:=create([-1,-1], eval(g_compts)):
> type(g,tensor_type);
> #print(g):
Build Christoffel I
> coord := [x[1], x[2], x[3], x[4]];
> D1g:=d1metric(g,coord): cf1:=Christoffel1 (D1g):
> #print(cf1);
> type(cf1,tensor_type); cf1_compts:=get_compts(cf1):
Build Christoffel II
> ginv:= invert (g, 'detg'):
> cf2:= Christoffel2 (ginv, cf1):  #Christoffel 2
> cf2_compts:=get_compts(cf2): type(cf2,tensor_type);
Build the spray coefficients G^i
> p:=prod(cf2,y_t,[2,1]): type(p,tensor_type);
> G:=prod(p,y_t,[2,1]): type(G,tensor_type);
> G1:=get_compts(G):
Build Jacobi matrix (G_y)
> man:=array(sparse,1..4,1..4):
> for i from 1 to 4 do for j from 1 to 4 do
>   man[i,j]:=simplify(diff(G1[i],y[j])); end do: end do:
> G_y:=create([1,-1],eval(man)): G_y1:=get_compts(G_y):
Build the first invariant (epsilon)
> eps:=prod(G_y,y_t,[2,1]):       eps1:=get_compts(eps):
> for i from 1 to 4 do epsilon[i]:=(eps1[i])/2-G1[i]; end do:
> epsilon:=create([1],
       array([epsilon[1],epsilon[2],epsilon[3],epsilon[4]])):
> epsilon1:=get_compts(epsilon);    # the first invariant
```

```
II-1 (g_x) - g'_x
> man:=array(sparse, 1..4, 1..4):
> for i from 1 to 4 do for j from 1 to 4 do
>   man[i,j]:=simplify(diff(G1[i],x[j])); end do: end do:
> G_x:=create([1,-1],eval(man)): G_x1:=get_compts(G_x):
II-2 (G_y_y)
> man:='man': man:=array(sparse,1..4,1..4,1..4):
> for i from 1 to 4 do for r from 1 to 4 do for j from 1 to 4 do
>   man[i,r,j]:=diff(G_y1[i,r],y[j]); end do; end do; end do:
> G_y_y:=create([1,-1,-1],eval(man)): G_y_y1:=get_compts(G_y_y):
II-3 (G_x_y)
> man:='man': man:=array(sparse,1..4,1..4,1..4):
> for i from 1 to 4 do for r from 1 to 4 do for j from 1 to 4 do
>   man[i,r,j]:=diff(G_x1[i,r],y[j]); end do; end do; end do:
> G_x_y:=create([1,-1,-1],eval(man)): G_x_y1:=get_compts(G_x_y):
> pr:=prod(y_t,G_x_y,[1,2]):  pr1:=get_compts(pr):
> prr:=prod(G_y,G_y,[2,1]): prr1:=get_compts(prr):
The second invariant (P)
> man:='man': man:=array(sparse,1..4,1..4):
> for i from 1 to 4 do for j from 1 to 4 do
>    man[i,j]:=simplify(-G_x1[i,j]+(pr1[i,j])/2+(prr1[i,j])/4);
> end do; end do:
> p:=create([1,-1],eval(man)): p1:=get_compts(p):
The last three invariants (R, B, D)
> for i from 1 to 4 do for j from 1 to 4 do for k from 1 to 4 do
    mant[i,j,k]:=diff(p1[i,j],x[k])-diff(p1[i,k],x[j]);
> end do: end do: end do:
> R:=create([1,-1,-1],eval(mant)): R1:=get_compts(R):
> for i from 1 to 4 do for j from 1 to 4 do
>   for k from 1 to 4 do for l from 1 to 4 do
>     manp[i,j,k,l]:=diff(R1[i,j,k],x[l]);
> end do: end do: end do: end do:
> B:=create([1,-1,-1,-1],eval(manp)): B1:=get_compts(B):
> for i from 1 to 4 do for j from 1 to 4 do
>   for k from 1 to 4 do for l from 1 to 4 do
>     manp[i,j,k,l]:=diff(G1[i],x[j],x[k],x[l]);
> end do: end do: end do: end do:
> D:=create([1,-1,-1,-1],eval(manp)): D1:=get_compts(D):
Particular cases
> for i from 1 to 4 do for j from 1 to 4 do
>  a(x[1],x[2],x[3],x[4])*y[i]*y[j]=a00[i,j]:
> end do: end do:
> for i from 1 to 4 do for j from 1 to 4 do
> p1[2,3]:=simplify(p1[2,3],
          {a(x[1],x[2],x[3],x[4])*y[i]*y[j]=a00[i,j]}):
> end do: end do:
#=====================================================================
```