The stability of tachyonic field system by factorization method

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Abstract. Using the associated Jacobi differential equation, we obtain the solution for the tachyonic potential. The stability of the tachyonic field is discussed by the factorization method. According to this method, we decompose the second order equation in terms of first order equations. The first order equations lead us to obtain the shape invariance condition.

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1 Introduction

The brane theory is a good candidate for the fundamental problem in High Energy Physics [7, 11, 21, 27], and the formulation of brane in terms of a single infinite extra dimension plays an important role in this field.

The brane scenario starts with the five dimensional space time with anti-de Sitter (AdS_5) with warped geometry, and the warped geometry in the brane theory described by the single real function and depends only bulk coordinate on the extra dimension. Also, the presence of scalar fields for the weak stabilizing brane depends only on the extra dimension ([4, 5, 6, 7, 11, 21, 23, 26, 27]).

As we know the standard cosmological model describes standard gravity with scalar curvature R, cosmological constant and non-relativistic dust-like matter.

Recently, the braneworld model is described by tachyon scalar field in five-dimensional space-time with AdS_5 geometry. On the other hand a number of authors have already demonstrated that the tachyon could play a useful role in cosmology [9, 10, 16, 17, 15, 24], independent of the fact that it is an unstable field. It can act as a source of dark matter and can lead to a period of inflation depending on the form of the associated potential.

Indeed it has been proposed as the source of dark energy for a particular class of potentials [1, 2, 3, 12, 18]. For example Sami & all ([25]) discussed the cosmological prospects of the rolling tachyon with exponential potential.

In this paper we obtain the potential which corresponds to the tachyon potential in presence of curvature R. This potential, in the special case complectly agree with Ref. [25].

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The stability of the system and the property of shape invariance are considered in correspondence with the Schrödinger equation by the tachyonic potential. We obtain the normal mode of the system by factorization method [8, 13, 14, 22]. But the shape invariance is just an integrability condition - an interesting feature in supersymmetric quantum mechanics, and the entire modes of system can be determined algebraically without ever refereing to underlying differential equations [23, 26].

The paper is organized as follows. In the next section we study the dynamics of the tachyon to solve the tachyon potential. The equations of motion are presented and the Einstein equations are discussed. By using the equation of motion and Einstein equation we obtain the warp factor, which leads us to obtain the corresponding potential for the tachyon field. In section 3, normal modes of tachyon system are obtained in the Schrödinger equation by the associated Jacobi differential equation. In section 4, we factorize the corresponding second order differential equation and obtain raising and lowering operators with respect to m ([19, 20]). A summary and an outlook are given in the conclusion.

2 The dynamics of the tachyon

The standard braneworld scenario is described by the real scalar field interaction with gravity via the usual Einstein Hilbert action, which has the general form,

$$S = \int d^4x dy \sqrt{g}\left(-\frac{1}{16\pi G}R + \mathcal{L}_T\right), \tag{2.1}$$

where R is the scalar curvature.

We now turn attention to the case of the tachyonic field. We consider that tachyons dominate in univese. In this case, the Lagrange density is given by,

$$\mathcal{L}_T = -V(T(y))\sqrt{1-\partial_\mu T\partial^\mu T}, \tag{2.2}$$

where T is the tachyon field, and $V(T)$ is the tachyonic field potential.

In this theory the line element is given by

$$dS_5^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{2.3}$$

where $e^{2A(y)}$ is the warp factor and $A(y)$ is the real function of the extra dimension, which gives rise to the warped geometry, and $\eta = diag(+---)$ describes the four-dimensional flat space-time with $\mu, \nu = 0, 1, 2, 3$. The geometry of the five dimensional space-time is then described by $A(y)$ and is driven by the extra coordinate y.

We consider the tachyon action and suppose that the tachyon field only depends on the extra dimension, that is, $T = T(y)$. The variation of the action with respect to the field leads to the following equation of motion,

$$T''(y) + \left(4A'(y)T'(y) + \frac{V_T}{V(T)}\right)(1 - T'^2) = 0, \tag{2.4}$$

where $V_T = \frac{dV}{dT}$.

The Einstein equation is then

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{2.5}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, the metric of the five dimensional is $g_{\mu\nu} = diag(e^{2A}, -e^{2A}, -e^{2A}, -e^{2A}, -1)$ and $T_{\mu\nu}$ is the energy-momentum tensor. We can simply consider $4\pi G = 1$ and $c = 1$. We are also using the following energy - momentum tensor corresponding to the $\mathcal{L}_T$ given by (2.2),

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}\mathcal{L})}{\delta g^{\mu\nu}}, \tag{2.6}$$

so, we can obtain the following equations,

$$\begin{aligned} T_{00} &= e^{2A(y)}V(T)\sqrt{1-T'^2} \\ T_{11} &= T_{22} = T_{33} = -e^{2A(y)}V(T)\sqrt{1-T'^2} \\ T_{44} &= \frac{-V(T)}{\sqrt{1-T'^2}}. \end{aligned} \tag{2.7}$$

The components of the Ricci tensor are,

$$\begin{aligned} R_{00} &= \left(4A'^2 + A''\right)e^{2A} \\ R_{11} &= R_{22} = R_{33} = -\left(4A'^2 + A''\right)e^{2A} \\ R_{\mu\nu} &= \eta_{\mu\nu}e^{2A}\left(4A'^2 + A''\right) \\ R_{44} &= -4\left(A'^2 + A''\right), \end{aligned} \tag{2.8}$$

thus, the Ricci scalar is given by,

$$R(y) = 20A'^2 + 8A'', \tag{2.9}$$

where R is in term of y.
We substitute the above results into equation (2.5) to obtain the two equations, using (2.5), (2.7) and (2.8) one can obtain,

$$\frac{1}{2}\left(4A'^2 + A''\right) - \frac{1}{4}R = V\sqrt{1-T'^2}, \tag{2.10}$$

and

$$2\left(A'^2 + A''\right) - \frac{1}{4}R = \frac{V}{\sqrt{1-T'^2}}. \tag{2.11}$$

The solution of the equations (2.10) and (2.11) lead us to obtain T'^2 and V,

$$T'^2 = \frac{6A''}{8(A'' + A'^2) - R}, \tag{2.12}$$

and

$$V = \frac{3}{2}A'\sqrt{R - A'^2}. \tag{2.13}$$

In the presence of the scalar field, we consider the specific case when R is constant. So that we can obtain $A'(y)$ from (2.9) as follows,

$$A'(y) = \frac{2}{5}\gamma\tanh(\gamma y), \tag{2.14}$$

where $\gamma = \frac{\sqrt{5R}}{4}$. By using (2.13) and (2.14), the corresponding potential for the tachyon field will be,

$$V(y) = \frac{12\gamma^2}{25}\tanh(\gamma y)\sqrt{5 - \tanh^2(\gamma y)}. \tag{2.15}$$

Now we draw V with respect to y, in that case this graph will be equivalent to the following equation,

$$V(y) = \eta \tanh(\gamma y), \tag{2.16}$$

so η will become the functional of R. The graphs of the potentials (2.15) and (2.16) are kink-like.

3 The stability of the system and the normal mode

The corresponding Schrödinger equation for the tachyonic potential (2.15) can be easily written as,

$$-\frac{d^2\Psi(y)}{dy^2} + V(y)\Psi(y) = k^2\Psi(y), \tag{3.1}$$

and

$$\frac{d^2\Psi(y)}{dy^2} + \left(k^2 - \eta\tanh(\gamma y)\right)\Psi(y) = 0, \tag{3.2}$$

With the definition of variable $x = \tanh(\gamma y)$ and $\Psi(x) = P^{\alpha,\beta}_{n,m}(x)U(x)$, we obtain the following Schrödinger equation,

$$\begin{aligned}&(1-x^2)P''_{n,m}(x)dx^2 + \left[2(1-x^2)\frac{U'(x)}{U(x)} - 2x\right]P'_{n,m}(x)\\ &+\left[(1-x^2)\frac{U''(x)}{U(x)} - 2x\frac{U'(x)}{U(x)} + \frac{k^2}{\gamma^2(1-x^2)} - \frac{\eta x}{\gamma^2(1-x^2)}\right]P_{n,m}(x) = 0,\end{aligned} \tag{3.3}$$

where $P^{\alpha,\beta}_{n,m}(x) = P_{n,m}(x)$.

Also here, in order to obtain the parameters, the eigenfunction and the normal mode for the tachyon potential, we compare (3.3) with the following associated Jacobi differential equation ([8, 13, 14, 22]),

$$\begin{aligned}&(1-x^2)P''_{n,m}(x) - [\alpha - \beta + (\alpha+\beta+2)x]\,P'_{n,m}\\ &+\left[n(\alpha+\beta+n+1) - \frac{m(\alpha+\beta+m+(\alpha-\beta))x}{1-x^2}\right]P_{n,m}(x) = 0.\end{aligned} \tag{3.4}$$

We compare (3.3) and (3.4), and obtain $U(x)$ and k^2 as follows,

$$U(x) = (-1)^{\frac{\alpha+\beta}{2}}(1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}}, \tag{3.5}$$

and

$$
\begin{aligned}
&k^2 = -\gamma^2 \left[m(\alpha + \beta + m) + \frac{(\beta - \alpha)^2}{4} \right] \\
&\eta = -\gamma^2(\beta - \alpha) \left[\frac{\alpha + \beta}{2} + m \right] \\
&\alpha + \beta = 0.
\end{aligned}
\tag{3.6}
$$

Finally, we obtain the solution of equation for the corresponding tachyon potential,

$$
\Psi(y) = (-1)^{\frac{\alpha+\beta}{2}} (1 - \tanh(\gamma y))^{\frac{\alpha}{2}} (1 + \tanh(\gamma y))^{\frac{\beta}{2}} \; P_{n,m}(\tanh(\gamma y)). \tag{3.7}
$$

As well, we infer the normal mode,

$$
k^2 = -\gamma^2(m^2 + \beta^2) = -\frac{5R}{16}(m^2 + \beta^2). \tag{3.8}
$$

We note that the stability of system is satisfied assuming the condition ($k^2 > 0$), so we have $R < 0$. In that case, for the negative R case (2.9) is written as,

$$
-R = 20A'^2 + 8A'', \tag{3.9}
$$

so that we can obtain $A'(y)$ in the form,

$$
A'(y) = -\frac{2}{5}\gamma \tanh(\gamma y). \tag{3.10}
$$

In that case the graph of the corresponding potential will be anti-kink.

4 The factorization method and shape invariance

As follows from [8, 13, 14, 22], any second order equation which has an exact solution or a normal mode can be factorized in terms of first order equations. These first order equations lead us to have raising and lowering operators. These first operators give us the shape invariance condition.

As we know, the associated Jacobi differential equation can be factorized by ladder and lower operators with respect to the parameter m as [13, 14],

$$
A^+_m(x)A^-_m(x)P_{n,m}(x) = E_{n,m}P_{n,m}(x), \tag{4.1}
$$

$$
A^-_m(x)A^+_m(x)P_{n,m-1}(x) = E_{n,m}P_{n,m-1}(x), \tag{4.2}
$$

where

$$
A^+_m(x) = \sqrt{1 - x^2}\frac{d}{dx} + \frac{(m - 1)x}{\sqrt{1 - x^2}}, \tag{4.3}
$$

$$
A^-_m(x) = -\sqrt{1 - x^2}\frac{d}{dx} + \frac{(\alpha - \beta) + (\alpha + \beta + m)x}{\sqrt{1 - x^2}}, \tag{4.4}
$$

$$E_{n,m} = (n - m + 1)(\alpha + \beta + n + m). \tag{4.5}$$

On the other hand, by comparing the Jacobi differential equation with the tachyonic equation we replace (3.6) into (4.3) and (4.4), and we get,

$$A^+_m(x) = \sqrt{1 - x^2}\frac{d}{dx} + \frac{(m - 1)x}{\sqrt{1 - x^2}}, \tag{4.6}$$

$$A^-_m(x) = -\sqrt{1 - x^2}\frac{d}{dx} + \frac{\eta + m^2\gamma^2 x}{m\gamma^2\sqrt{1 - x^2}}. \tag{4.7}$$

If we express the above equations with respect to y, so we obtain

$$A^+_m(y) = \frac{1}{\sqrt{1 - \tanh^2(\gamma y)}}\left[\frac{1}{\gamma}\frac{d}{dy} + (m - 1)\tanh(\gamma y)\right], \tag{4.8}$$

$$A^-_m(y) = \frac{1}{\sqrt{1 - \tanh^2(\gamma y)}}\left[-\frac{1}{\gamma}\frac{d}{dy} + \frac{\eta}{m\gamma^2} + m\tanh(\gamma y)\right]. \tag{4.9}$$

We remark that (4.8) and (4.9) are first order equations and correspond to (3.2). It is interesting to link these first order operators to the generators of the $N = 2$ algebra.

5 Conclusions

In this paper, we have discussed the tachyon field and obtained the corresponding potential in the braneworld scenario. We solved the corresponding potential in the case of constant curvature. Also, we calculated the energy spectrum (normal mode) and bound states of energy and obtained the stability of system in the case of negative curvature. Using the factorization method, we derived some laddering operators. The investigation of the shape invariance condition will be an interesting problem in future.

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