Solving some functional and operational equations

Alina Olteanu and Octav Olteanu

Abstract. In the present work we recall some improved versions of some general results first published in [14], concerning the functional equation $g = g \circ f$ (g is a given function or operator, f being the "unknown" function, respectively operator). Further, we solve a special functional equation, when $g(x) = x^{-1}d^x, x > 0, d \in \mathbb{R}, d > 1$. The corresponding solution f has interesting special qualities related to prime positive integers. The related operational equation is solved.

M.S.C. 2000: 26A06, 26A09, 26A48, 26A51, 26D07, 26D15, 47A60, 47A63, 11A41, 11D61.
Key words: functional equations, functions of a real variable, decreasing functions with interesting properties vis-a-vis to primes, operational equations related to convex operators.

1 Introduction

If g is a C^n function, $n \geq 1$, which satisfies some conditions required by the implicit function theorem for $G(x, y) := g(x) - g(y)$ at $(x_0, y_0), x_0 \neq y_0$, then the equation

$$g(x) = g(f(x)), \ x \in V_{x_0} \tag{1.1}$$

has a C^n nontrivial solution in a neighborhood of x_0. There is a special case of equation (1.1), when $g : A \subset X \to X$, ($A = A_l \cup A_r$, where A_l, A_r are convex subsets of an order-complete vector lattice X) , g is a convex operator on a convex set C containing A, which satisfies some conditions (see [14, Theorem 1.10] or its version from [10]-[13] and [15]), when the solution f is a decreasing function (operator) $f : A \to A$, such that $f^{-1} = f$ and the unique fixed point of f is the minimum point of g.

Moreover, some formulae (with a geometric meaning) can be used to "construct" the solution f (see Theorems 2.1 and its abstract operatorial version [14, Theorem 1.10]). These formulae involve only the order structure of an (order) complete vector lattice.

In some applications, one uses order complete vector lattices which are also commutative Banach algebras of operators. To verify the general-type conditions from [14, Theorem 1.10] for such special spaces X and special operators g, one uses differential calculus, convexity and functional calculus related to inequalities (Theorem 3.2 of the present work).

Applied Sciences, Vol.10, 2008, pp. 193-198.

In the real scalar case ($g : I \subset \mathbb{R} \to \mathbb{R}$), more information on f are obtained (Theorems 2.1, 3.1). Such type results were first published in [14], and then improved and applied in [10]-[13] and in [15]. For an approach to the complex case, see [13].

2 General results

Many analytic elementary functions satisfy (on some intervals $I \subset \mathbb{R}$) the hypothesis of the following theorem. In [14, Fig.1], the geometric meaning of equation (2.1) is clear.

Theorem 2.1. *Let* $u, v \in \overline{\mathbb{R}},\ u < v,\ a \in]u, v[$; *let* $g :]u, v[\to \mathbb{R}$ *be a continuous function. Assume that*:

(a) *there exist* $\lim\limits_{x \downarrow u} g(x) = \lim\limits_{x \uparrow v} g(x) = \lambda \in \overline{\mathbb{R}}$;

(b) g *"decreases" (strictly) in the interval* $]u, a]$, *and is (strictly) increasing in the interval* $[a, v[$.

Then there exists a strictly decreasing function $f :]u, v[\to [u, v[$, *such that*

$$g(x) = g(f(x)), \ \forall\, x \in]u, v[\tag{2.1}$$

and f *has the following qualities*:

(i) $\lim\limits_{x \downarrow u} f(x) = v,\ \lim\limits_{x \uparrow v} f(x) = u$;

(ii) a *is the unique fixed point of* f;

(iii) $f^{-1} = f$ *in* $]u, v[$;

(iv) f *is continuous in* $]u, v[$; (v) *if* $g \in C^n(]u, v[\setminus \{a\}), n \in \mathbb{Z}_+ \cup \{\infty\}$, *then* $f \in C^n([u, v[\setminus \{a\})$;

(vi) *if* g *is derivable in* $]u, v[\setminus \{a\}$, *so is* f;

(vii) *if* $f \in C^1(]a - \varepsilon, a + \varepsilon[$ *(for an* $\varepsilon > 0$), *then* $f'(a) = -1$;

(viii) *if* g *is analytic at* a,*then* f *is derivable at* a *and* $f'(a) = -1$;

(ix) *if* $g \in C^3(]u, v[\setminus \{a\}), g''(a) \neq 0$, *and there exist* $\rho_1 := \lim\limits_{x \to a} f'(x)$, $\rho_2 := \lim\limits_{x \to a} f''(x) \in \mathbb{R}$, *then* $f \in C^2(]u, v[) \cap C^3(]u, v[\setminus \{a\})$ *and* $f''(a) = -\dfrac{2}{3} \cdot \dfrac{g^{(3)}(a)}{g''(a)}$;

(x) *let* $g_r := g\,\big|_{[a,v[}$, $g_\ell := g\big|_{]u,a]}$; *then we have the following constructive formulae for* $f(x)$:

$$f(x_0) = \left(g_r^{-1} \circ g_\ell\right)(x_0) = \sup\{x \in [a, v[; g_r(x) \le g_\ell(x_0)\},\ \forall\, x_0 \in]u, a],$$

$$f(x_0) = \left(g_\ell^{-1} \circ g_r\right)(x_0) = \inf\{x \in]u, a]; g_\ell(x) \le g_r(x_0)\},\ \forall\, x_0 \in [a, v[.$$

In Section 3 we will apply the "abstract operational" version of Theorem 2.1, [14, Theorem 1.10].

3 Applications

The next result is partially obtained from Theorem 2.1, applied to $g(x) = x^{-1}d^x,\ x > 0,\ d > 1$. This special function g and (2.1) lead to special qualities of the solution , such as those mentioned at 7) and 6) from below.

Theorem 3.1. *Let* $d \in]1,\infty[$. *There exists a (strictly) decreasing function* $f :]0,\infty[\to]0,\infty[$, *such that*

$$\frac{d^x}{x} = \frac{d^{f(x)}}{f(x)}, \ \forall\, x > 0,$$

and f *has the following qualities:*

1) $\lim\limits_{x\downarrow 0} f(x) = \infty, \ \lim\limits_{x\uparrow\infty} f(x) = 0$;

2) $a := \dfrac{1}{\ln d}$ *is the unique fixed point of* f;

3) $f^{-1} = f$ in $]0,\infty[$; 4) $f \in C^{\infty}\left(]0,\infty[\setminus \dfrac{1}{\ln d}\right)$ *and there exists* $f'\left(\dfrac{1}{\ln d}\right) = -1$;

5) *we have the following formulae for* $f(x)$:

$$f(x_0) = \sup\left\{x \geq \frac{1}{\ln d}; \frac{d^x}{x} \leq \frac{d^{x_0}}{x_0}\right\}, \ \forall x_0 \in]0, \frac{1}{\ln d}],$$

$$f(x_0) = \inf\left\{x \in]0, \frac{1}{\ln d}]; \frac{d^x}{x} \leq \frac{d^{x_0}}{x_0}\right\}, \ \forall x_0 \geq \frac{1}{\ln d};$$

6) *the asymptotic behaviour of* f *is given by:*

$$\lim_{x\to\infty} \frac{f(x)\, d^x}{x} = \lim_{x\to 0} \frac{x d^{f(x)}}{f(x)} = 1;$$

7) *if* $d, x \in \mathbb{Z}, \ d, x \geq 2$ *are prime numbers, then* $f(x)$ *is a rational number if and only if* $d = x = 2$ *and, in this case, we have* $f(2) = 1$.

The next result is partially an application of [14, Theorem 1.10], but also of some results related to bounded self-adjoint operators on Hilbert spaces.

Let H be a Hilbert space, and $\mathcal{A} = \mathcal{A}(H)$ the real vector space of all linear bounded self-adjoint operators acting on H. Let $T \in \mathcal{A}(H)$. Define

$$\mathcal{A}_1 = \mathcal{A}_1(T) = \{A \in \mathcal{A}(H); AT = TA\}$$

$$X := X(T) := \{U \in \mathcal{A}_1; UA = AU, \ \forall A \in \mathcal{A}_1\}$$

$$X_+ := \{U \in X; \langle U(h), h\rangle \geq 0, \ \forall h \in H\}.$$

It is known that X is an order-complete vector lattice and a commutative algebra of operators (see [2, pp. 303-305]).

Theorem 3.2. *Let* $d \in]1,\infty[$,

$$a_1 := \frac{1}{\ln d}, A_\ell := \{U \in X : \sigma(U) \subset]0, a_1[\} \cup \{a_1 I\}$$

$$A_r := \{U \in X; \sigma(U) \subset]a_1, \infty[\} \cup \{a_1 I\},$$

where $\sigma(U)$ *is the spectrum of* U, *and* $I : H \to H$ *is the identity operator.*

Denote $a := a_1 I \in A_\ell \cap A_r$ *and* $A := A_\ell \cup A_r$.

Then there exists a strictly decreasing map $F : A \to A$, *such that*

$$U^{-1} d^U = [F(U)]^{-1} d^{F(U)}, \ \forall\, U \in A$$

and F has the following properties:

(i) $a = a_1 I$ *is the unique fixed point of* F;

(ii) F *is invertible and* $F^{-1} = F$ *on* A;

(iii) F *can be "constructed" by formulae:*

$$F(U_0) = \sup\left\{U \in A_r; U^{-1}d^U \leq U_0^{-1}d^{U_0}\right\}, \ \forall\, U_0 \in A_\ell,$$

$$F(U_0) = \inf\left\{U \in A_\ell; U^{-1}d^U \leq U_0^{-1}d^{U_0}\right\}, \ \forall\, U_0 \in A_r.$$

Proof. We apply [14, Theorem 1.10] to $X, a = a_1 I = \dfrac{1}{\ln d} I, A_\ell, A_r, A$ defined above and to $g : A \to X$,

$$\begin{aligned} g(U) &:= U^{-1}d^U = U^{-1}\exp(U \ln d) = \\ &= U^{-1} + \sum_{n=1}^{\infty} \frac{U^{n-1}}{n!}(\ln d)^n. \end{aligned}$$

In [14, pp. 79-80], we proved that $U \longmapsto U^n, n \geq 1$, is convex on X_+. To prove that $g_\ell := g\,|_{A_\ell}, g_r := g\,|_{A_r}$ are convex, it is sufficient to prove that g is convex in $C := \{U \in X; \sigma(U) \subset]0,\infty[\} \supset A$. To do this, since $\displaystyle\sum_{n=1}^{\infty} \frac{U^{n-1}}{n!}(\ln d)^n$ is convex as a supremum of convex operators, it remains to show that $U \longmapsto U^{-1}$ is convex in C.

Let $U_1, U_2 \in C$, $\lambda \in [0,1]$. We have to show that

$$[(1-\lambda)U_1 + \lambda U_2]^{-1} \leq (1-\lambda)U_1^{-1} + \lambda U_2^{-1}. \tag{3.1}$$

Since U_1, U_2 are positive and invertible and X is a commutative algebra, the inequality (3.1) is equivalent to:

$$\left[(1-\lambda)U_1U_2^{-1} + \lambda I\right]^{-1} \leq (1-\lambda)U_1^{-1}U_2 + \lambda I = (1-\lambda)\left(U_1U_2^{-1}\right)^{-1} + \lambda I, \lambda \in [0,1].$$

Thus we have to prove that

$$[(1-\lambda)U + \lambda I]^{-1} \leq (1-\lambda)U^{-1} + \lambda I, \lambda \in [0,1], U \in C. \tag{3.2}$$

Let $u \in]0,\infty[$ and $\lambda \in [0,1]$. By the convexity of the elementary function $x \longmapsto x^{-1}$ in $]0,\infty[$, it follows that:

$$[(1-\lambda)u + \lambda]^{-1} \leq (1-\lambda)u^{-1} + \lambda, \lambda \in [0,1], u \in]0,\infty[. \tag{3.3}$$

Since $\sigma(U) \subset]0,\infty[$, we can integrate the last inequality (3.3) on $\sigma(U)$, with respect to the spectral measure associated to U, which is positive. Thus (3.3) leads to (3.2), which is equivalent to (3.1), so that the convexity of g_ℓ, g_r is proved.

On the other hand we have:

$$[g'(U)](V) = \left[-U^{-2}d^U + U^{-1}d^U \ln d\right] \cdot V = U^{-2}d^U\,[U \ln d - I] \cdot V, U \in A, V \in X.$$

If $U \in A_\ell \setminus \{a_1 I\}$, then: $\sigma(U) \subset]0, \dfrac{1}{\ln d} I[\Rightarrow 0 < U < \dfrac{1}{\ln d} I = a \Rightarrow U \ln d - I < 0 \Rightarrow (U \ln d - I)V \leq 0, \forall V \in X_+$.

We also have: $\sigma\left(U\ln d-I\right)=\varphi\left(\sigma\left(U\right)\right)$, where $\varphi\left(t\right)=t\ln d-1<0,\forall t\in\sigma\left(U\right)$. Hence $\sigma\left(U\ln d-I\right)=\sigma\left(\varphi\left(U\right)\right)=\varphi\left(\sigma\left(U\right)\right)\subset]-\infty,0[$.

This yields: $U\ln d-I$ is invertible and $\sigma\left[\left(U\ln d-I\right)^{-1}\right]=\left[\sigma\left(U\ln d-I\right)\right]^{-1}\subset\,]-\infty,0[$. Thus $\left(U\ln d-I\right)^{-1}<0$ and we have

$$g'\left(U\right)\in-\mathrm{Izom}_{+}\left(X\right),\forall U\in A_{\ell}\backslash\left\{a_1I\right\}.$$

Similarly, $g'\left(U\right)\in\mathrm{Izom}_{+}\left(X\right),\forall U\in A_{r}\backslash\left\{a_1I\right\}$.

To finish the proof we only have to verify the last condition of [14, Theorem 1.10]:

$$R\left(g_{\ell}\right)=R\left(g_{r}\right).$$

Let $g_{\ell}\left(U_1\right)\in R\left(g_{\ell}\right),U_1\in A_{\ell}\backslash\left\{a_1I\right\}$. Let $f:]0,\infty[\to]0,\infty[$ be the function of Theorem 3.1. Let

$$U_2:=F\left(U_1\right):=\int\limits_{\sigma\left(U_1\right)}f\left(t\right)d_{E_{U_1}}\left(t\right),$$

where $d_{E_{U_1}}$ is the spectral measure attached to U_1. Using the fact that f is strictly decreasing and $a_1=\dfrac{1}{\ln d}$ is the fixed point of f,it is clear that f applies $]0,\dfrac{1}{\ln d}[$ onto $]\dfrac{1}{\ln d},\infty[$.

Using this, one obtains :

$$\sigma\left(U_2\right)=\sigma\left(F\left(U_1\right)\right)=f\left(\sigma\left(U_1\right)\right)\subset]\frac{1}{\ln d},\infty[,$$

because of: $U_1\subset A_{\ell}\backslash\left\{a_1I\right\}$ is equivalent to $\sigma\left(U_1\right)\subset]0,\dfrac{1}{\ln d}[$. Thus $U_2\in A_r\backslash\left\{a_1I\right\}$. We have

$$g_{\ell}\left(u_1\right)=g_r\left(f\left(u_1\right)\right),\forall u_1\in\sigma\left(U_1\right)\subset]0,\frac{1}{\ln d}[.$$

By Theorems 2.1, 3.1, and integrating on $\sigma\left(U_1\right)$, with respect to $d_{E_{U_1}}$, we obtain:

$$g_{\ell}\left(U_1\right)=g_r\left(F\left(U_1\right)\right)=g_r\left(U_2\right)\in R\left(g_r\right).$$

Thus $R\left(g_{\ell}\right)\subset R\left(g_r\right)$. Similarly, $R\left(g_r\right)\subset R\left(g_{\ell}\right)$. Applying [14, Theorem 1.10], the conclusion follows. □

References

[1] I. Colojoară, *Elements of Spectral Theory* (in Romanian), Ed. Academiei, Bucharest, 1968.

[2] R. Cristescu, *Ordered Vector Spaces and Linear Operators*, Ed. Academiei and Abacus Press, Tunbridge Wells, Kent, 1976.

[3] R. Cristescu, Gh. Grigore, C. Niculescu, G. Păltineanu, N. Popa and D.T. Vuza, *Order Structures in Functional Analysis* (in Romanian), Vol. 3, Ed. Academiei Române, Bucharest, 1992.

[4] C. Drăguşin, O. Olteanu and M. Gavrilă, *Mathematical Analysis. Theory and Applications* (in Romanian), Vol. II, Ed. MATRIX ROM, Bucharest, 2007.

[5] C. Dumitrescu and O. Olteanu, *The inverse of a convex operator. Newton's method for convex operators*, UPB Sci. Bull, Mechanical Engineering, 53, 3-4 (1991), 23-29.

[6] Gh. Gussi, *Itinerary in Mathematical Analysis* (in Romanian), Lyceum 95, Ed. Albatros, Bucharest, 1970.

[7] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers.* Fifth Edition, Oxford University Press Inc., New York, 1979.

[8] M. Nicolescu and O. Olteanu, *Differential Calculus. Mathematical Analysis* (*first part*) (in Romanian), Ed. Printech, Bucharest, 1999.

[9] V. Olariu and O. Olteanu, *Mathematical Analysis* (in Romanian), Ed.Semne, Bucharest, 1999.

[10] A. Olteanu and O. Olteanu, *Some exponential-type functional equations solved by a geometrical general method*, U.P.B. Sci. Bull. Ser. A, 67, 1 (2005), 35-48.

[11] A. Olteanu and O. Olteanu, *Algebraic functional equations solved by a geometrical geneal method.* In Proc. of the 3-rd International Colloquium "Mathematics in Engineering and Numerical Physics", October 7-9, 2004, Bucharest, 208-217. Balkan Society of Geometers. Geometry Balkan Press, 2005.

[12] A. Olteanu and O. Olteanu, *Solving some special functional equations by a general geometrical method*, Annals of University of Craiova, 33 (2006), 54-66.

[13] A. Olteanu and O. Olteanu, *Solving some special functional equations by a general "geometrical" method and an approach to the complex case*, Rev. Roumaine Math. Pures Appl. 51, 5-6 (2006), 735-745.

[14] O. Olteanu and Gh. Simion, *A new geometric aspect of the implicit function principle and Newton's method for operators*, Math. Reports 5(55), 1 (2003), 61-84.

[15] O. Olteanu and C. Radu, *Solving some functional and operational equations by a general constructive method*, U.P.B. Sci. Bull., Series A, 69, 4 (2007), 57-68.

[16] G. Păltineanu, *Mathematical Analysis* (in Romanian), Ed. AGIR, Bucharest, 2002.

[17] W. Rudin, *Real and Complex Analysis* (in Romanian), Theta Eds., Bucharest, 1999.

[18] F.-H. Vasilescu, *Initiation in Linear Operator Theory* (in Romanian), Ed. Tehnică, Bucharest, 1987.

Authors' addresses:

Alina Olteanu
Department of Computer Science, University of Alabama,
101 Houser Hall Box, 870290 Tuscaloosa, Al 35487-0290, USA.
E-mail: aoltenu@cs.ua.edu

Octav Olteanu
Department of Mathematics-Informatics I, Faculty of Applied Sciences,
University Politehnica of Bucharest, 313 Splaiul Independenţei,
RO-060042 Bucharest, Romania.
E-mail: olteanuoctav@yahoo.ie