A note on (k, f, l)-chordal polygons

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Abstract. In this article the results concerning the existence of (k, f, l)-chordal polygons that appeared in [2, 3] are generalized by using a theorem presented in [1]. Furthermore, it is pointed out that there is an error in a theorem stated in [2] by using a counterexample.

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1 Introduction

The concept of (k, f, l)-chordal polygons was introduced in [2], and can be regarded as a generalization of k-chordal polygons. A special category of (k, f, l)-chordal polygons, the so-called (k, λ, l)-chordal polygons, was analyzed in [3]. Throughout the article we use the same definitions and notation as in [2, 3].

Let $\alpha_1, \dots, \alpha_n$ be positive reals. A k-chordal polygon with sides $\alpha_1, \dots, \alpha_n$ is denoted $\underline{A} = A_1A_2 \dots A_n$. If we consider the central angles $\theta_i = \angle A_iCA_{i+1}$ for $i = 1, \dots, n$ and $A_1 \equiv A_{n+1}$ (where C is the center of the circum-circle), then for a k-chordal polygon it holds that $\sum_{i=1}^{n} \theta_i = 2k\pi$. This means that the total arc of the polygon is k times the circumference of the circle.

The definition of a (k, f, l)-chordal polygon in [2] is that a k-chordal polygon with sides $\alpha_1, \dots, \alpha_n$ exists and an l-chordal polygon with sides $f(\alpha_1), \dots, f(\alpha_n)$ also exists (f is a positive real function). Our interest is to provide necessary and sufficient conditions for the existence of a (k, f, l)-chordal polygon.

The article is divided as follows: In Section 2 we present our main results which are direct conclusions of a theorem that was proved in [1]. In Section 3 we note that there is an error in Theorem 2.2 appeared in [2], by using the results from Section 2 and a proper counterexample.

2 Existence results for (k, f, l)-chordal polygons

Throughout the article we consider k-chordal polygons with sides $\alpha_1, \dots, \alpha_n$ which are positive reals and l-chordal polygons with sides $f(\alpha_1), \dots, f(\alpha_n)$ which are also positive reals. We let $\alpha^* = \max_{1 \le i \le n} \alpha_i$, $\hat{\alpha} = \min_{1 \le i \le n} \alpha_i$ and $f^* = \max_{1 \le i \le n} f(\alpha_i)$.

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The next Theorem stated in [1] provides a necessary and sufficient condition for the existence of a k-chordal polygon:

Theorem 1. *For each $k = 1, \dots, m$ there exists a k-chordal polygon with sides $\alpha_1, \dots, \alpha_n$, if and only if*

$$\sum_{i=1}^{n} \arcsin\left(\frac{\alpha_i}{\alpha^*}\right) > m\pi. \tag{2.1}$$

Since Theorem 1 gives a necessary and sufficient condition for the existence of a k-chordal polygon, it is easy to prove the following Theorem for the existence of a (k, f, l)-chordal polygon.

Theorem 2. *Let $\alpha_1, \dots, \alpha_n$ be positive reals and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a real function. A (k, f, l)-chordal polygon exists if and only if*

$$\sum_{i=1}^{n} \arcsin\left(\frac{\alpha_i}{\alpha^*}\right) > k\pi, \tag{2.2}$$

and

$$\sum_{i=1}^{n} \arcsin\left(\frac{f(\alpha_i)}{f^*}\right) > l\pi. \tag{2.3}$$

Proof. Using Theorem 1 for the k-chordal polygon with sides $\alpha_1, \dots, \alpha_n$ and also for the l-chordal polygon with sides $f(\alpha_1), \dots, f(\alpha_n)$ the proof is complete. □

If in addition we consider that f is an increasing (decreasing) real function, we can get directly the following corollary.

Corollary 1. *Let $\alpha_1, \dots, \alpha_n$ be positive reals and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing (decreasing) real function. A (k, f, l)-chordal polygon exists if and only if*

$$\sum_{i=1}^{n} \arcsin\left(\frac{\alpha_i}{\alpha^*}\right) > k\pi,$$

and

$$\sum_{i=1}^{n} \arcsin\left(\frac{f(\alpha_i)}{f(\alpha^*)}\right) > l\pi, \qquad \left(\sum_{i=1}^{n} \arcsin\left(\frac{f(\alpha_i)}{f(\hat{\alpha})}\right) > l\pi\right). \tag{2.4}$$

Obviously, we note that the existence of a (k, f, l)-chordal polygon implies the existence of every (j, f, i)-chordal polygon for each $j = 1, ..., k$ and $i = 1, ..., l$.

The conditions in Theorem 2 and Corollary 1 are necessary and sufficient for the existence of a (k, f, l)-chordal polygon. This means that other results can be variations of these. In fact Theorems 2.1 and 2.3 in [2] can be proved directly by using the present results. On the other hand Theorem 2.2 in [2] has an error and does not hold. This observation is described in the next Section.

3 An error in a Theorem

In order to stress the error we use a counterexample.

Let $\alpha_1 = 10$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 9$, where $n = 5$ and $\alpha^* = \alpha_1 = 10$. Also let $k = 2$ and we will initially see that the assumptions of Theorem 2.2 in [2] are satisfied.

Firstly, we have that

$$\sum_{i=1}^{5} \alpha_i = 46 > 2k\alpha^* = 40,$$

thus the property $\sum_{i=1}^{n} \alpha_i > 2k\alpha^*$ of Theorem 2.2 in [2] is satisfied.

Secondly, let $f(x) = x^2$ which is a convex, monotonously increasing real function in $\mathbb{R}_+$ with $f(0) = 0$. Thus, assumptions 1 and 2 of Theorem 2.2 are also satisfied.

Finally, the third assumption is satisfied for $l = 1$ since:

$$\arcsin \frac{f(2k\alpha^*/n)}{f(\alpha^*)} = \arcsin \frac{64}{100} \simeq 0.6944983 > \frac{l\pi}{n} = \frac{\pi}{5} \simeq 0.6283185.$$

Since the property and all the assumptions of Theorem 2.2 in [2] are satisfied a $(j, f, 1)$-chordal polygon should exist for $j = 1, 2$ (remember that $k = 2$ and $l = 1$). This means that a 2-chordal pentagon must exist for $\alpha_1 = 10$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 9$. On the other hand, inequality (2.1) from Theorem 1 or equivalently inequality (2.2) from Theorem 2 does not hold since:

$$\sum_{i=1}^{5} \arcsin\left(\frac{\alpha_i}{10}\right) \simeq 6.0498744 < k\pi = 2\pi \simeq 6.2831853.$$

This means that the specific 2-chordal pentagon does not exist and consequently Theorem 2.2 in [2] is false. Note that a similar argument was used in [1] in order to disprove a posed conjecture.

Next, we stress that the 1-chordal pentagon (or simply a cyclic convex pentagon) with sides $f(\alpha_1), \ldots, f(\alpha_5)$ of our counterexample truly exists since inequality (2.3) from Theorem 2 holds:

$$\sum_{i=1}^{5} \arcsin\left(\frac{f(\alpha_i)}{100}\right) \simeq 5.3474048 > l\pi \simeq 3.1415927.$$

The reason for the existence of this 1-chordal pentagon is that assumption 3 of Theorem 2.2 is a sufficient condition for the existence of the l-chordal polygon. This is true because the following inequality holds for a convex increasing real function $f : \mathbb{R}_+ \to \mathbb{R}_+$ when $\sum_{i=1}^{n} \alpha_i > 2k\alpha^*$:

$$\sum_{i=1}^{n} \arcsin\left(\frac{f(\alpha_i)}{f(\alpha^*)}\right) \geq n \arcsin \frac{f(2k\alpha^*/n)}{f(\alpha^*)}.$$

Since the above inequality holds, equation (2.4) from Corollary 1 implies that assumption 3 of Theorem 2.2 is indeed a sufficient condition for the existence of the l-chordal polygon.

On the other hand the property $\sum_{i=1}^{n} \alpha_i > 2k\alpha^*$ that appears in Theorem 2.2 in [2], is a necessary and not a sufficient condition for the existence of a k-chordal polygon with sides $\alpha_1, \ldots, \alpha_n$. This is the reason why the 2-chordal pentagon of our counterexample does not exist. The authors in [2] do not provide additional assumptions to support the existence of the k-chordal polygon. Since the property $\sum_{i=1}^{n} \alpha_i > 2k\alpha^*$ is not a sufficient condition for the existence of a k-chordal polygon, it is concluded that the assumptions of Theorem 2.2 are not sufficient conditions for the existence of a (k, f, l)-chordal polygon. Consequently, Theorem 2.2 in [2] is finally false.

References

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