The polar moment of inertia of the projection curve

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Abstract. H. R. Müller, [2], had studied the polar moment of inertia for the orbit curves during the one-parameter closed planar motions and the area of the projection curve, [3], under the spatial kinematics. In this paper, under the one-parameter closed homothetic motion in three dimensional Euclidean space, we expressed the polar moment of inertia of the projection curve of the closed space curve. We also obtained formulas equivalent to the results given by [2] and [4].

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§1. Introduction

A one-parameter homothetic (equiform) motion of a rigid body in n-dimensional Euclidean space is given analytically by, [5],

$$x' = hAx + C \tag{1.1}$$

in which x' and x are the position vectors, represented by column matrices, of a point X in the fixed space R' and the moving space R respectively; A is an orthogonal matrix, C a translation vector and h is the homothetic scale of the motion. Also, h, A and C are continuously differentiable functions of a parameter t, which may be identified with time. Without any loss of generality we may suppose that for $t = 0$ the origins in R and R' coincide, so for $t = 0$, $h = 1$, $A = I$ and $C = 0$.

In this study, we consider the one-parameter closed homothetic motion in Euclidean 3-space. Since the motion is closed, all the quantities depending on the parameter t are the periodic functions of the same period T. Let $\{O; e_1, e_2, e_3\}$ and $\{O'; {e_1}', {e_2}', {e_3}'\}$ be two right-handed sets of orthonormal vectors that are rigidly linked to the moving space R and fixed space R', respectively, and denote E, E' the matrices

$$E = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad E' = \begin{bmatrix} {e_1}' \\ {e_2}' \\ {e_3}' \end{bmatrix}.$$

Then we may write

$$E = AE' \quad \text{or} \quad E' = A^t E, \tag{1.2}$$

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where A is a positive orthogonal 3×3-matrix and the superscript "t" indicates the transpose. Since $A \in SO(3)$ we may write

$$AA^t = I,$$

where I is the unit matrix. This equation, by differentiation with respect to t, yields

$$dA.A^t + A.dA^t = 0,$$

which shows that the matrix

$$\Omega = dA.A^t$$

is antisymmetric. We may write

$$\Omega = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix},$$

where w_i, $i = 1, 2, 3$, are the linear differential forms with respect to t, i.e. $w_i = f_i(t)dt$. Differentiation of (1.2) with respect to t yields

$$dE = \Omega E$$

or

$$de_i = w_k e_j - w_j e_k, \quad (i, j, k = 1, 2, 3; 2, 3, 1; 3, 1, 2). \tag{1.3}$$

Then, we may write

$$de_i = \boldsymbol{\omega} \times e_i,$$

where

$$\boldsymbol{\omega} = w_1 e_1 + w_2 e_2 + w_3 e_3$$

is called the rotation vector of the motion and " $\times$ " denotes the vector product.

§2. The polar moment of inertia of the projection curve

I.

Let X be a fixed point in R with

$$\vec{OX} = x = x_1 e_1 + x_2 e_2 + x_3 e_3.$$

If we denote

$$\vec{OO'} = u = u_1 e_1 + u_2 e_2 + u_3 e_3,$$

for the position vector of X in R' we may write

$$\vec{O'X} = x' = hx - u. \tag{2.1}$$

The point X describes a closed curve (X), its *path*, in R' during the one-parameter closed homothetic motion. The projection of this closed path, in the direction of a fixed unit vector e', on any plane P is a closed curve, say (X^n). We suppose that the curve (X^n) is uniformly covered with the mass elements

$$dm = ||\boldsymbol{\omega^n}||dt = |cos\theta|\omega dt,$$

where $\omega = ||\boldsymbol{\omega}||$ is the instantaneous angular velocity of the motion, $\boldsymbol{\omega^n}$ is the normal component of $\boldsymbol{\omega}$ to the plane P and $\theta = \theta(t)$ is the angle between the vectors $\boldsymbol{\omega}$ and e'.

For the projection of x' in the direction of e' on P, we have

$$x^n = x' - \langle e', x'\rangle e', \tag{2.2}$$

where x^n is the position vector of the projection point X^n of $X' \in R'$ and "$\langle , \rangle$" denotes the scalar product. Thus, the polar moment of inertia of the curve (X^n) with respect to the origin O' of R' $(O' \in P)$ is

$$M_X = \oint ||x^n||^2 dm, \tag{2.3}$$

where the integration is taken along the closed curve (X^n).

If we substitute (2.2) into (2.3), for the polar moment of inertia of the projection curve of (X) we obtain

$$M_X = \int_0^T \left\{||x'||^2 - \langle e', x'\rangle^2\right\} |cos\theta|\omega dt. \tag{2.4}$$

Let the direction of projection in R' is given by the unit vector $e' = a_1 e_1 + a_2 e_2 + a_3 e_3$. Then, if we substitute (2.1) into (2.4), we get

$$M_X = M_O + \rho \sum_{i=1}^{3} x_i^2 - \sum_{i,j=1}^{3} b_{ij} x_i x_j + \sum_{i=1}^{3} c_i x_i, \tag{2.5}$$

where

$$\rho = \int_0^T h^2 |cos\theta|\omega dt, \quad b_{ij} = \int_0^T h^2 a_i a_j |cos\theta|\omega dt,$$

$$c_i = 2\int_0^T h\left(a_i \sum_{j=1}^{3} a_j u_j - u_i\right)|cos\theta|\omega dt$$

and M_O is the polar moment of inertia of the projection curve of the orbit curve (O).

Now, we choose the moving coordinate system such that $b_{ij} = 0$ for $i \neq j$. Then,

$$b_{ii} = \int_0^T h^2 a_i^2 |cos\theta|\omega dt = \int_0^T h^2(t) a_i^2(t) |cos\theta(t)| \sqrt{\sum_{i=1}^{3} f_i^2(t)} dt.$$

If we use the mean-value theorem of the integral calculus, we may write

$$b_{ii} = h^2(t_{1i}) a_i^2(t_{1i})\sigma, \quad t_{1i} \in [0, T], \quad i = 1, 2, 3, \tag{2.6}$$

and also

$$\rho = h^2(t_4)\sigma, \quad t_4 \in [0,T], \tag{2.7}$$

where $\sigma = \int_0^T |cos\theta|\omega dt$. In this case, from (2.5) we have

$$M_X = M_O + \sigma \sum_{i=1}^{3} \left(h^2(t_4) - h^2(t_{1i})a_i^2(t_{1i})\right) x_i^2 + \sum_{i=1}^{3} c_i x_i. \tag{2.8}$$

On the other hand, since $a_i^2(t) \leq 1$, we may write $b_{ii} \leq \rho$, i.e.

$$h^2(t_{1i})a_i^2(t_{1i}) \leq h^2(t_4).$$

Let the maximum value of the function $h^2(t)$ be B in the interval $[0,T]$, i.e., $0 \leq h^2(t) \leq B$ for all $t \in [0,T]$. Then, we have

$$0 \leq h^2(t_4) - h^2(t_{1i})a_i^2(t_{1i}) \leq B, \quad i = 1,2,3.$$

If we use the continuity of the function $h^2(t)$, we get

$$h^2(t_4) - h^2(t_{1i})a_i^2(t_{1i}) = h^2(t_{2i}), \quad t_{2i} \in [0,T].$$

Thus, we can rewrite (2.8) as

$$M_X = M_O + \sigma \sum_{i=1}^{3} h^2(t_{2i})x_i^2 + \sum_{i=1}^{3} c_i x_i$$

or there exists at least one point $t_0 \in [0,T]$ such that

$$M_X = M_O + \sigma h^2(t_0) \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} c_i x_i. \tag{2.9}$$

We may give the following theorem:

Theorem 1. *Let us consider the 1-parameter closed motions of Euclidean 3-space. If the projection curves (in the direction of a unit vector) of closed point paths have equal polar moment of inertia, then such points lie on the same sphere with the center*

$$C = \left(\frac{-c_1}{2h^2(t_0)\sigma}, \frac{-c_2}{2h^2(t_0)\sigma}, \frac{-c_3}{2h^2(t_0)\sigma}\right).$$

Different spheres (having same center C) correspond to different values of M_X.

II.

Let X and Y be two fixed points in R and Z be another point on the line segment XY, that is,

$$z_i = \lambda x_i + \xi y_i, \quad \lambda + \xi = 1.$$

Using (2.9), we get

$$M_Z = \lambda^2 M_X + 2\lambda\xi M_{XY} + \xi^2 M_Y, \tag{2.10}$$

where

$$M_{XY} = M_O + \sigma h^2(t_0)\sum_{i=1}^{3} x_i y_i + \frac{1}{2}\sum_{i=1}^{3} c_i(x_i + y_i) \tag{2.11}$$

is called the *mixture polar moment of inertia* of the projection curves of (X) and (Y). It is clearly seen that $M_{XY} = M_{YX}$ and $M_{XX} = M_X$.

Since

$$M_X - 2M_{XY} + M_Y = h^2(t_0)\sigma d_{XY}^2, \tag{2.12}$$

we can rewrite (2.10) as follows:

$$M_Z = \lambda M_X + \xi M_Y - \lambda\xi h^2(t_0)\sigma d_{XY}^2, \tag{2.13}$$

where d_{XY} is the distance between the points X and Y. By the orientation of the line XY we will distinguish $d_{XY} = -d_{YX}$. Since X, Y and Z are collinear, we may write

$$d_{XZ} + d_{ZY} = d_{XY}.$$

Thus, if we denote

$$\lambda = \frac{d_{ZY}}{d_{XY}} = \frac{b}{d}, \quad \xi = \frac{d_{XZ}}{d_{XY}} = \frac{a}{d},$$

from (2.13) we get

$$M_Z = \frac{1}{d}(bM_X + aM_Y) - h^2(t_0)\sigma ab. \tag{2.14}$$

The equivalent result for planar kinematics is given by Müller, [2].

Now, we consider that the points X and Y trace the same closed space curve. In this case, for the projection curves in the direction of e' we have $M_X = M_Y$. Then, from (2.14) we obtain

$$M_X - M_Z = h^2(t_0)\sigma ab. \tag{2.15}$$

Thus, we have Holditch-type result[1], [1], for the polar moment of inertia of projection curves. The equivalent result for planar kinematics is given by [2].

So, we may give the following theorem:

Theorem 2. *Let us consider a line segment with the constant length. If the end-points of the line segment trace the same space curve in R', then a different point on this segment traces another space curve. The difference between the polar moments of inertia of the projection curves (in the direction of a unit vector) of these space*

[1]The classical Holditch Theorem: If the endpoints X, Y of a segment of fixed length are rotated once on an oval, then a given point Z of this segment, with $\overline{XZ} = a$, $\overline{ZY} = b$, describes a closed, not necessarily convex, curve. The area of the ring-shaped domain bounded by the two curves is πab.

curves depends on not only the distances of the chosen point from the endpoints but also the homothetic scale.

III.

Let $X_1 = (x_i), X_2 = (y_i)$ and $X_3 = (z_i)$, i=1,2,3 be noncollinear points in R and $Q = (q_i)$ be a point on the plane determined by X_1, X_2, X_3, (Fig. 1). Then, we may write

$$q_i = \lambda_1 x_i + \lambda_2 y_i + \lambda_3 z_i, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

If we use (2.9), we obtain

$$M_Q = \lambda_1^2 M_{X_1} + \lambda_2^2 M_{X_2} + \lambda_3^2 M_{X_3} + 2\lambda_1\lambda_2 M_{X_1X_2} + 2\lambda_1\lambda_3 M_{X_1X_3} + 2\lambda_2\lambda_3 M_{X_2X_3}.$$

After eliminating the mixture polar moments of inertia by using (2.12), we get

$$\begin{aligned} M_Q &= \lambda_1 M_{X_1} + \lambda_2 M_{X_2} + \lambda_3 M_{X_3} \\ &\quad -h^2(t_0)\sigma\left(\lambda_1\lambda_2 d^2_{X_1X_2} + \lambda_1\lambda_3 d^2_{X_1X_3} + \lambda_2\lambda_3 d^2_{X_2X_3}\right). \end{aligned} \tag{2.16}$$

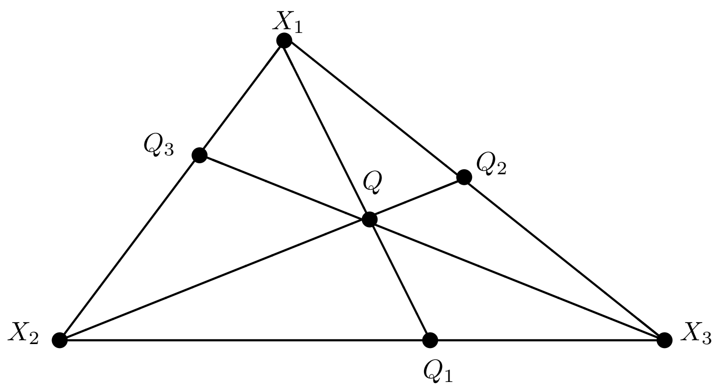

Fig. 1

On the other hand, if we consider the point $Q_1 = (s_i)$, we may write

$$s_i = \xi_1 y_i + \xi_2 z_i, \quad q_i = \xi_3 x_i + \xi_4 s_i, \quad \xi_1 + \xi_2 = \xi_3 + \xi_4 = 1.$$

Thus, we have $\lambda_1 = \xi_3,\ \lambda_2 = \xi_1\xi_4,\ \lambda_3 = \xi_2\xi_4$, i.e.

$$\lambda_1 = \frac{d_{QQ_1}}{d_{X_1Q_1}}, \quad \lambda_2 = \frac{d_{X_1Q}\ d_{Q_1X_3}}{d_{X_1Q_1}\ d_{X_2X_3}}, \quad \lambda_3 = \frac{d_{X_1Q}\ d_{X_2Q_1}}{d_{X_1Q_1}\ d_{X_2X_3}}.$$

Similarly, considering the points Q_2 and Q_3, respectively, we find

$$\lambda_i = \frac{d_{QQ_i}}{d_{X_iQ_i}} = \frac{d_{X_jQ}\ d_{X_kQ_j}}{d_{X_jQ_j}\ d_{X_kX_i}} = \frac{d_{X_kQ}\ d_{Q_kX_j}}{d_{X_kQ_k}\ d_{X_iX_j}}, \quad i,j,k = 1,2,3(cyclic).$$

Then, from (2.16) the generalization of (2.14) is found as

$$M_Q = \sum \frac{d_{QQ_i}}{d_{X_iQ_i}} M_{X_i} - h^2(t_0)\sigma \sum \left(\frac{d_{X_kQ}}{d_{X_kQ_k}}\right)^2 d_{Q_kX_j} d_{X_iQ_k}. \tag{2.17}$$

If X_1, X_2, X_3 trace the same closed space curve, then the difference between the polar moments of inertia is

$$M_{X_1} - M_Q = h^2(t_0)\sigma \sum \left(\frac{d_{X_k Q}}{d_{X_k Q_k}} \right)^2 d_{Q_k X_j} d_{X_i Q_k}.$$

Then, we can give the following theorem:

Theorem 3. *Let us consider a triangle with the vertices X_1, X_2 and X_3 in R. Let the vertices of the triangle trace the same space curve in R'. Then, the point Q on the plane determined by X_1, X_2, X_3 traces another space curve. The difference between the polar moments of inertia of the projection curves of these space curves depends on the distances of the moving triangle and the homothetic scale h.*

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