On the ordered sets in n-dimensional real inner product spaces

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Abstract. Let X be a real inner product space of dimension ≥ 2. In [2], W. Benz proved the following theorem for $x, y \in X$ with $x < y$: "The Lorentz-Minkowski distance between x and y is zero (i.e., $l(x, y) = 0$) if and only if $[x, y]$ is ordered". In this paper, we obtain necessary and sufficient conditions for Lorentz-Minkowski distances $l(x, y) > 0$, $l(x, y) < 0$ with the help of ordered sets in n-dimensional real inner product spaces.

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1 Introduction

Let X be a n-dimensional real inner product space, i.e., a real vector space furnished with an inner product

$$g : X \times X \longrightarrow \mathbb{R}, \quad g(x, y) = xy$$

satisfying $xy = yx,\ x(y + z) = xy + xz,\ \alpha(xy) = (\alpha x)y,\ x^2 > 0$ (for all $x \neq 0$ in X) for all $x, y, z \in X$, $\alpha \in \mathbb{R}$. Moreover X need not be complete, i.e., that X need not be a real Hilbert space. We assume that the dimension of X is $n \geq 2$ and t be a fixed element of X satisfying $t^2 = 1$ and define

$$t^{\perp} := \{x \in X : tx = 0\}$$

Then clearly $t^{\perp} \oplus \mathbb{R}t = X$. For any $x \in X$, there are uniquely determined $\overline{x} = x - x_0 t \in t^{\perp}$ and $x_0 = tx \in \mathbb{R}$ with

$$x = \overline{x} + x_0 t$$

Definition 1. The *Lorentz-Minkowski distance* of $x, y \in X$ defined by the expression

$$l(x, y) = (\overline{x} - \overline{y})^2 - (x_0 - y_0)^2$$

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Definition 2. If the mapping $\varphi : X \to X$ preserves the Lorentz-Minkowski distance for each $x, y \in X$ then φ is called *Lorentz transformation.*

Under all the translation functions, Lorentz-Minkowski distances remain invariant and it might be noticed that the theory does not seriously depend on the chosen t, for more details we refer [1].

2 Ordered Sets in n-Dimensional Real Inner Product Spaces

Let x, y be elements of a n-dimensional real inner product space X $(n \geq 2)$, and define a relation on X such that

$$x \leq y \Leftrightarrow l\,(x, y) \leq 0 \text{ and } x_0 \leq y_0$$

Observe that an element of X can not be comparable to other element of X, for example neither $e \leq 0$ nor $0 \leq e$ if we take e from $t^{\perp}$. Therefore, $(X, \leq)$ is a partially ordered set but not completely ordered set. For the properties of "$\leq$ ", see [2].

Definition 3. Let we take two elements of $x, y \in X$ satisfying $x < y$ $(x \leq y, x \neq y)$ and define

$$[x, y] = \{z \in X : x \leq z \leq y\}\,.$$

$[x, y]$ is called ordered if and only if,

$$u \leq v \text{ or } v \leq u$$

holds true for all $u, v \in [x, y]$.

In [2], W. Benz proved the following theorem and it will be the foundation of this paper.

Theorem 4. *Let $x, y \in X$ with $x < y$, then $l\,(x, y) = 0$ if and only if $[x, y]$ is ordered.*

Corollary 5. *Let $x, y \in X$ with $x \neq y$, then $l\,(x, y) = 0$ if and only if either $[x, y]$ or $[y, x]$ ordered.*

Proof. First we assume $l\,(x, y) = 0$ and this implies $(\overline{x} - \overline{y})^2 - (x_0 - y_0)^2 = 0$ and it is clear that $x_0 \neq y_0$. Thus we get either $x_0 < y_0$ or $y_0 < x_0$. Let we take $x_0 < y_0$ and we get $[x, y]$ ordered and similarly if we take $y_0 < x_0$ then obtain $[y, x]$ ordered. The second part of Corollary 5 immediately follows from the Definition 3 and Theorem 4. □

Lemma 6. *Let $x, y \in X$ with $x < y$ and $[x, y]$ be an ordered set and take $u \in X$. Then u is an element of $[x, y]$ if and only if $[x, u]$ and $[u, y]$ are ordered sets.*

Proof. Firstly, assume $u \in [x, y]$. Therefore, from [2], there is an element α of $\mathbb{R}$ (actually $\alpha \in [0, 1] \subset \mathbb{R}$) such that

$$u := x + \alpha\,(y - x)$$

satisfied. Clearly $l\,(x, u) = 0 = l(u, y)$ and $x_0 < u_0 < y_0$.
Conversely, we want to show $u \in [x, y]$. Because of $[x, u]$ and $[u, y]$ are ordered sets, we obtain $x \leq u \leq y$ and this yields $u \in [x, y]$. □

Lemma 7. *$\forall x, y, z \in X,\ [x, y],\ [y, z]$ be ordered sets. Then the set $[x, z]$ is ordered if and only if the set $\{y - x, z - x\}$ is linear dependent.*

Proof. Let we assume firstly, $[x, z]$ be ordered. Then clearly $y \in [x, z]$ and there is a real number $\alpha \in \mathbb{R}$ such that

$$y := x + \alpha (z - x)$$

holds. Therefore, $y - x = \alpha (z - x)$ and this implies $\{y - x, z - x\}$ is linear dependent. Conversely, if $\{y - x, z - x\}$ is linear dependent, there is one $\lambda \in \mathbb{R}$ such that

$$z := x + \lambda (y - x)$$

and thus we get $l (x, z) = 0$. □

The proof of the following lemmas are not difficult.

Lemma 8. *$\forall x, y, z \in X$ with $x < y$ and $[x, z],\ [y, z]$ be ordered sets. Then the set $[x, y]$ is ordered if and only if the set $\{y - x, z - x\}$ is linear dependent.*

Lemma 9. *Let X be an n-dimensional real inner product space and $[x, y],\ [z, k]$ be ordered sets in X. Then*

$$[x, y] \cap [z, k] = [r, s],\ \{m\},\ \phi,$$

i.e., the intersection set of ordered sets may be an ordered set or a set which consists of a unique element, or empty set.

2.1 Positive Lorentz-Minkowski Distances ($l (x, y) > 0$)

Theorem 10. *Let X be a n-dimensional real inner product space ($n \geq 2$) and x, y be elements of X with $x \neq y$ and $x_0 \leq y_0$. Then followings are equivalent.*

- *(i)* $l (x, y) > 0$.
- *(ii) There is at least one $s \in X - \{x, y\}$ such that $[x, s],\ [y, s]$ are ordered while $[x, y]$ is not ordered.*
- *(iii) There is at least one $k \in X - \{x, y\}$ such that $[k, x],\ [k, y]$ are ordered while $[x, y]$ is not ordered.*

Proof. For all elements of $x, y \in X$, it is not hard to see the equality

$$l (x, y) = l (0, y - x)$$

Therefore, instead of considering x and y, we can prove the theorem with respect to 0 and $y - x$. Firstly we take an orthonormal basis of X as follows:

$$\theta := \{t, e_1, e_2, \ldots, e_{n-1}\}$$

$l (x, y) > 0$ implies $l (0, y - x) > 0$. Thus, for $i \in \{1, 2, \ldots, n - 1\}$ there are λ_i and $\mu \in \mathbb{R}$ (uniquely determined) such that

$$y - x := \mu t + \sum_{i=1}^{n-1} \lambda_i e_i \tag{2.1}$$

holds.

$(i) \Rightarrow (ii)$ $l(x,y) > 0$ implies $\sum_{i=1}^{n-1} \lambda_i^2 - \mu^2 > 0$, and $x_0 \leq y_0$ implies $\mu \geq 0$. Define

$$u := \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n-1} \lambda_i^2} + \mu\right) t + \sum_{i=1}^{n-1} \frac{\lambda_i}{2}\left(1 + \frac{\mu}{\sqrt{\sum_{i=1}^{n-1} \lambda_i^2}}\right) e_i$$

Then we get $l(0,u) = 0 = l(y-x,u)$, i.e., $[0,u]$, $[y-x,u]$ are ordered sets. Clearly $[0,y-x]$ is not ordered since $l(x,y) > 0$. To prove that (ii) holds, translate 0 to x, then we get $[x,x+u]$, $[y,u+x]$ are ordered sets and $[x,y]$ is not an ordered set.

$(ii) \Rightarrow (i)$ In order to prove (i), it is enough to show $l(x,y) \nleq 0$. We suppose $l(x,y) \leq 0$. Clearly $x < y$ and this implies $x < y < s$. Firstly we assume $l(x,y) < 0$ and take

$$u := \frac{1}{2}(x+y) \text{ and } v := \frac{1}{2}(x+s)$$

Observe $l(x,u) < 0$, $l(x,v) = 0$, $l(u,v) = 0$ and moreover $u_0 < v_0$. It is immediately follows from the fact $l(u,v) = 0$, obtain $[u,v]$ is ordered. Obviously $u \in [x,v]$. In order to see that we observe $x_0 < u_0 < v_0$ and $l(x,u)$, $l(u,v) \leq 0$. But $u \notin [x,v]$ since $l(x,u) \neq 0$ and this is a contradiction. Obviously $l(x,y) \neq 0$, otherwise; $[x,y]$ would be ordered. Therefore, we obtain $l(x,y) > 0$.

$(i) \Rightarrow (iii)$ $l(x,y) > 0$ implies $\sum_{i=1}^{n-1} \lambda_i^2 - \mu^2 > 0$, and $x_0 \leq y_0$ implies $\mu \geq 0$. Define

$$v := \frac{1}{2}\left(-\sqrt{\sum_{i=1}^{n-1} \lambda_i^2} + \mu\right) t + \sum_{i=1}^{n-1} \frac{\lambda_i}{2}\left(1 - \frac{\mu}{\sqrt{\sum_{i=1}^{n-1} \lambda_i^2}}\right) e_i$$

Then we get $l(v,0) = 0 = l(v,y-x)$, i.e., the sets $[v,0]$, $[v,y-x]$ are ordered sets. Clearly $[0,y-x]$ is not ordered since $l(x,y) > 0$. To show that (iii) holds, translate 0 to x, then we get $[v+x,x]$, $[v+x,y]$ are ordered sets, but $[x,y]$ is not ordered.

$(iii) \Rightarrow (i)$ We suppose $l(x,y) \leq 0$. Clearly $x < y$ and this implies $k < x < y$. Firstly we assume $l(x,y) < 0$ and take

$$u' := \frac{1}{2}(x+y) \text{ and } v' := \frac{1}{2}(x+k)$$

Observe $l(x,u') < 0$, $l(v',x) = 0$, $l(v',u') = 0$, i.e., $[v',u']$ is ordered and moreover $v'_0 < u'_0$. Obviously $x \in [v',u']$ and this yields $l(x,u') = 0 = l(v',x)$, and this is a contradiction. Therefore, $l(x,y) \nless 0$. Obviously $l(x,y) \neq 0$, otherwise; $[x,y]$ would be ordered, so we obtain $l(x,y) > 0$.

□

Remark 11. *In the previous theorem u and v are not unique if* $\dim X \geq 3$*. Indeed if we take X as 3-dimensional real standard inner-product space, i.e., $X = \mathbb{R}^3$ and take*

$$t := (0, 0, 1), \qquad y - x := \left(\frac{3}{2}, 0, 0\right),$$

then we obtain

$$u' := \left(\frac{3}{4}, \frac{\sqrt{7}}{4}, 1\right) \neq \frac{3}{4}t + \frac{3}{4}e_1 =: u$$

$$v' := \left(\frac{3}{4}, -\frac{\sqrt{7}}{4}, 1\right) \neq -\frac{3}{4}t + \frac{3}{4}e_1 =: v$$

Proof of the following theorem is not difficult.

Theorem 12. *Let X be a n-dimensional real inner product space ($n \geq 2$) and x, y be elements of X with $x \neq y$ and $x_0 < y_0$. $l(x, y) > 0$ if and only if there is not any element of X such that $[x, s]$, $[s, y]$ are ordered sets.*

2.2 The Case $l(x, y) = 0$

Theorem 13. *Let X be a n-dimensional real inner product space ($n \geq 2$) and x, y be elements of X with $x \neq y$ and $x_0 < y_0$. Then followings are equivalent.*

(i) $l(x, y) = 0$.

(ii) There are at least $m, s \in X - \{x, y\}$ such that the sets $[x, s]$, $[y, s]$, $[m, x]$, $[m, y]$, $[m, s]$ are ordered sets.

Proof.

$(i) \Rightarrow (ii)$ At first, assume $l(x, y) = 0$, i.e., $l(0, y - x) = 0$ and we represent $y - x$ as before (2.1). This implies $[0, y - x]$ is ordered, and $\sum_{i=1}^{n-1} \lambda_i^2 - \mu^2 = 0$, $0 < y_0 - x_0$. If we take $u, v \in X$ such as

$$u := \varsigma_1 \left(\mu t + \sum_{i=1}^{n-1} \lambda_i e_i\right) \qquad , \varsigma_1 > 1$$

$$v := \varsigma_2 \left(\mu t + \sum_{i=1}^{n-1} \lambda_i e_i\right) \qquad , \varsigma_2 < 0,$$

then we get

$$l(0, u) = l(y - x, u) = l(v, 0) = l(v, y - x) = 0$$

A simple calculation shows that $l(v, u) = 0$, i.e, $[v, u]$ is a ordered set. If we translate 0 to x, we get the sets $[x, u + x]$, $[y, u + x]$, $[v + x, x]$, $[v + x, y]$, $[v + x, u + x]$ are ordered. Thus we could find $s, m \in X$.

$(ii) \Rightarrow (i)$ Conversely, for suitable $s, m \in X$, we assume that $[x,s]$, $[y,s]$, $[m,x]$, $[m,y]$, $[m,s]$ are ordered. Clearly

$$m \leq x \leq y \leq s$$

and thus $x \in [m,s]$ and $y \in [m,s]$, so this implies that there are real numbers $\alpha, \beta \in [0,1]$ such that

$$x := m + \alpha (s - m)$$
$$y := m + \beta (s - m)$$

then this yields $l(x,y) = 0$, i.e., $[x,y]$ is ordered.

□

Remark 14. *It is not hard to see that these elements m, s are not unique even if the dimension of X is two.*

2.3 Negative Lorentz-Minkowski Distances ($l(x,y) < 0$)

Theorem 15. *Let X be a n-dimensional real inner product space ($n \geq 2$) and x, y be elements of X with $x \neq y$ and $x_0 < y_0$. Then $l(x,y) < 0$ if and only if there is at least $s \in X$ such that $[x,s]$, $[s,y]$ are ordered but $[x,y]$ is not ordered.*

Proof. Assume firstly $l(x,y) < 0$ and similarly to previous proofs, we consider the elements $0, y - x$ instead of x, y. $l(x,y) < 0$ yields $\sum_{i=1}^{n-1} \lambda_i^2 - \mu^2 < 0$. If we may choose the element $u \in X$ such as:

$$u := \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n-1} \lambda_i^2} + \mu\right) t + \sum_{i=1}^{n-1} \frac{\lambda_i}{2}\left(1 + \frac{\mu}{\sqrt{\sum_{i=1}^{n-1} \lambda_i^2}}\right) e_i$$

Then we get $l(0,u) = 0 = l(u, y - x)$, i.e., the sets $[0,u]$, $[u, y - x]$ are ordered sets. Let we take $s \in X$ such that $[x,s]$ and $[s,y]$ are ordered but $[x,y]$ is not ordered. Using the property of "$\leq$" $x \leq s,\ s \leq y$ we obtain $x \leq y$, i.e., $l(x,y) < 0$. □

Remark 16. *In Theorem 15 the element s is not unique even if* $\dim X = 2$. *In fact for the case $n = 2$,*

$$s := \frac{\lambda_1 + \mu}{2} e + \frac{\lambda_1 + \mu}{2} t + x$$
$$s' := \frac{\lambda_1 - \mu}{2} e + \frac{-\lambda_1 + \mu}{2} t + x$$

are different elements of X and Theorem 15 holds for s, s'.

Proofs of the following theorems can be easily proved by the preceding proofs.

Theorem 17. *Let X be a n-dimensional real inner product space $(n \geq 2)$ and x, y be elements of X with $x \neq y$ and $x_0 < y_0$. If $l(x, y) < 0$ then there is not any element of X such that $[x, s]$, $[y, s]$ are ordered sets.*

Theorem 18. *Let X be a n-dimensional real inner product space $(n \geq 2)$ and x, y be elements of X with $x \neq y$ and $x_0 < y_0$. If $l(x, y) < 0$ then there is not any element of X such that $[s', x]$, $[s', y]$ are ordered sets.*

References

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