Random system of lines in the Euclidean plane E_2

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Abstract. In this paper we consider a random variable arising from a problem of geometrical intersection between a fixed convex body K and a system of random lines in E_2.

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1 Introduction

Let E_2 be the Euclidean plane and let K be a convex non empty and bounded domain of area S_K and with boundary ∂K of length L. We consider a family F of random, uniformly distributed n-lines $\{G_1, ..., G_n\}$ with $n \geq 2$. We assume that if $G_h, G_k \in F$, then $G_h \cap G_k \neq \varnothing$. It is possible that this points belongs to K or not. In this way we have a random variable $X_{(n,K)}$. In this paper we give the following result

Theorem 1. *The expression of the mean value* $\mathbf{E}\left(X_{(n,K)}\right)$*, the k-moments* $\mathbf{E}\left(X^k_{(n,K)}\right)$ *and the variance* $\sigma^2\left(X_{(n,K)}\right)$ *of the random variable* $X_{(n,K)}$ *can be calculated as follows*

$$\mathbf{E}\left(X_{(n,K)}\right) = \alpha\pi\frac{S_K}{L^2}, \qquad \mathbf{E}\left(X^k_{(n,K)}\right) = \left[\sum_{J_1+...+J_\alpha=k}\frac{k!}{J_1!...J_\alpha!}\right]\frac{\pi S_K}{L^2},$$

$$\sigma^2\left(X_{(n,K)}\right) = \frac{\pi S_K}{L^2}\left(1 - \frac{\pi S_K}{L^2}\right)\alpha^2,$$

where

$$\alpha = \frac{n\,(n-1)}{2},$$

and k is a positive integer.

Other results about the computation of the variance are investigated in [2] and an extension in the 3-dimensional Euclidean space of the same problem is studied in [6].

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2 Main Results

Let N be the set of natural numbers, $n \geq 2$ a fixed natural integer, $\{G_1, ..., G_n\}$ and K as in introduction. We can state the following

Theorem 2. *Let us consider the random variable* $X_{(n,K)}$. *Then*

$$\mathbf{E}\left(X_{(n,K)}\right) = \alpha\pi\frac{S_K}{L^2},$$

where $\alpha = \frac{n(n-1)}{2}$, L *is the length of* ∂K *and* S_K *is the area of the domain defined by* K.

Proof. It is easy to see that, denoting with L the length of ∂K and dG the elementary measure of the lines in the Euclidean plane E_2, we have

$$\int_{\left\{G\cap K\neq\varnothing\right\}} dG = L.$$

Since $G_1, ..., G_n$ are stochastically independent, we get

$$\int_{\left\{G\cap K\neq\varnothing\right\}} dG_1 \wedge ... \wedge\ dG_n = L^n.$$

If we consider the lines $G_1, ..., G_n$, then the intersection points might belong to K or not. Hence we obtain a random variable which we denote by $X_{(n,K)}$.

In order to compute the variance, we have the following integral

$$\int_{\left\{G\cap K\neq\varnothing\right\}} X_{(n,K)} dG_1 \wedge ... \wedge\ dG_n$$

We define the application $\epsilon_{hk} = 1$ if $G_h \cap G_k \in K$ (with $h \neq k = 1, ..., n$) and zero otherwise. Then

$$X_{(n,K)} = \sum_{h,k=1} \epsilon_{hk}.$$

Further, let us consider

$$I_2 := \int_{\left\{G_h, G_k\cap K\neq\varnothing\right\}} \epsilon_{hk} dG_h \wedge dG_k.$$

If $(G_h \cap E_k) \in K$, then we have $\epsilon_{hk} = 1$. We denote with λ_k the chord intercepted by G_k on K (and its length).

We have

$$\int_{\left\{G_h, G_k\cap K\neq\varnothing\right\}} \epsilon_{hk} dG_h \wedge dG_k = \int_{\left\{G_k\cap K\neq\varnothing\right\}} \left(\int_{\left\{G_k\cap K\neq\varnothing\right\}} dG_h\right) dG_k,$$

but it is well known that

$$\int_{\left\{G_h\cap\lambda_k\neq\varnothing\right\}} dG_h = \lambda_k,$$

$$\int_{\left\{ G_k \cap K \neq \emptyset \right\}} \lambda_k dG_k = \pi S_K.$$

Moreover,

$$\int_{\left\{ G\cap K\neq \emptyset \right\}} X_{(n,K)} dG_1 \wedge ... \wedge \ dG_n = \int_{\left\{ G\cap K\neq \emptyset \right\}} \sum_{h,k=1} \epsilon_{hk} dG_1 \wedge ... \wedge \ dG_n =$$

$$\pi S_K L^{n-2} + ... + \pi S_K L^{n-2}.$$

Taking into account that the number of the different sets $\{G_h, G_k\}$ is (the binomial coefficient)

$$\alpha = \frac{n\,(n-1)}{2},$$

we have

$$\int_{\left\{ G\cap K\neq \emptyset \right\}} X_{(n,K)} dG_1 \wedge ... \wedge \ dG_n = \alpha \pi S_K L^{n-2}.$$

By definition,

$$\mathbf{E}\left(X_{(n,K)}\right) := \frac{\int_{\left\{ G\cap K\neq \emptyset \right\}} X_{(n,K)} dG_1 \wedge ... \wedge \ dG_n}{\int_{\left\{ G\cap K\neq \emptyset \right\}} dG_1 \wedge ... \wedge \ dG_n},$$

and hence $mathbf E\left(X^2_{(n,K)}\right) = \alpha\pi \frac{S_K}{L^2}$. □

Now we compute

$$\mathbf{E}\left(X^2_{(n,K)}\right) = \frac{\int_{\left\{ G\cap K\neq \emptyset \right\}} X^2_{(n,K)} dG_1 \wedge ... \wedge \ dG_n}{\int_{\left\{ G\cap K\neq \emptyset \right\}} dG_1 \wedge ... \wedge \ dG_n}.$$

We put

$$J = \int_{\left\{ G\cap K\neq \emptyset \right\}} X^2_{(n,K)} dG_1 \wedge ... \wedge \ dG_n$$

We can prove that

$$X^2_{(n,K)} = \left(\sum_{h,k=1} \epsilon_{hk} \right)^2,$$

and then

$$X^2_{(n,K)} = \sum_{h,k=1} \epsilon^2_{hk} + 2 \sum_{(h,k)\neq(s,n)} \epsilon_{hk}\epsilon_{sn}.$$

With this observations we can compute the integral

$$J = \int_{\left\{ G\cap K\neq \emptyset \right\}} \left(\sum_{h,k=1} \epsilon^2_{hk} + 2 \sum_{(h,k)\neq(s,n)} \epsilon_{hk}\epsilon_{sn} \right) dG_1 \wedge ... \wedge \ dG_n.$$

Considering that

$$\int_{\left\{G\cap K\neq\varnothing\right\}}\epsilon_{hk}^{2}dG_{h}\wedge dG_{k}=\int_{\left\{G_{k}\cap K\neq\varnothing\right\}}\left(\int_{\left\{G_{h}\cap\lambda_{k}\neq\varnothing\right\}}dG_{h}\right)dG_{k},$$

where λ_k is the chord intercepted by G_k on K, we obtain

$$J=\pi S_K L^{n-2}+...+\pi S_K L^{n-2}+2\pi S_K L^{n-2}+...+2\pi S_K L^{n-2}.$$

Taking into account that the number of different sets $\{G_h, G_k\}$ is α and that $G_1, ..., G_2$ are independent, we infer

$$\mathbf{E}\left(X_{(n,K)}^{2}\right)=\frac{\pi S_K\alpha L^{n-2}}{L^n}=\frac{\pi S_K\alpha^2}{L^2}.$$

We obtain

Theorem 3. *Let us consider the random variable $X_{(n,K)}$. Hence*

$$\sigma^2\left(X_{(n,K)}\right)=\frac{\pi S_K}{L^2}\left(1-\frac{\pi S_K}{L^2}\right)\alpha^2,$$

where $\alpha=\frac{n(n-1)}{2}$, L is the length of ∂K and S_K is the area of the domain defined by K.

Moreover, we note that

$$X_{(n,K)}^{k}=\left(\sum_{h,k=1}\epsilon_{hk}\right)^{k},$$

implies

$$\mathbf{E}\left(X_{(n,K)}^{k}\right)=\left[\sum_{J_1+...+J_\alpha=k}\frac{k!}{J_1!...J_\alpha!}\right]\frac{\pi S_K}{L^2}.$$

3 Applications

1. As first case we consider in the plane a square Q of side a. We can compute the following values

$$\mathbf{E}\left(X_{(n,Q)}\right)=\frac{\alpha\pi}{16}\approx 0,196345\alpha.$$

$$\mathbf{E}\left(X_{(n,Q)}^{k}\right)=\left[\sum_{J_1+...+J_\alpha=k}\frac{k!}{J_1!...J_\alpha!}\right]\frac{\pi}{16}.$$

and the variance is

$$\sigma^2 X_{(n,Q)}=\frac{\pi}{16}\left(1-\frac{\pi}{16}\right)\alpha^2\approx 0,15779\alpha^2.$$

2. Taking in plane a rectangle R of sides a and b we have

$$\mathbf{E}\left(X_{(n,R)}\right) = \frac{\alpha\pi ab}{4\left(a+b\right)^2},$$

$$\mathbf{E}\left(X^k_{(n,R)}\right) = \left[\sum_{J_1+...+J_\alpha=k} \frac{k!}{J_1!...J_\alpha!}\right] \frac{\pi}{4\left(a+b\right)^2},$$

and the variance

$$\sigma^2 X_{(n,Q)} = \frac{\pi}{4\left(a+b\right)^2}\left(1-\frac{\pi}{4\left(a+b\right)^2}\right)\alpha^2.$$

3. Let C be a circle of radius δ. We have

$$\mathbf{E}\left(X^k_{(n,C)}\right) = \frac{\alpha}{4},$$

$$\mathbf{E}\left(X^k_{(n,C)}\right) = \left[\sum_{J_1+...+J_\alpha=k} \frac{k!}{J_1!...J_\alpha!}\right] \frac{\alpha}{4}$$

and the variance is

$$\sigma^2\left(X_{(n,C)}\right) = \frac{\pi}{4}\left(1-\frac{\alpha}{4}\right)\alpha^2 \approx 0,16855\alpha^2.$$

4. As last case we consider an equilateral triangle T of side a, obtaining,

$$\mathbf{E}\left(X^k_{(n,T)}\right) = \frac{\alpha\pi\sqrt{3}}{18} \approx 0,3023\alpha,$$

$$\mathbf{E}\left(X^k_{(n,T)}\right) = \left[\sum_{J_1+...+J_\alpha=k} \frac{k!}{J_1!...J_\alpha!}\right] \frac{\pi\sqrt{3}}{18},$$

and for the variance the expression

$$\sigma^2\left(X_{(n,T)}\right) = \frac{\pi\sqrt{3}}{18}\left(1-\frac{\pi\sqrt{3}}{18}\right)\alpha^2 \approx 0,2109\alpha^2.$$

Remark 4. *We observe that in examples 1, 3, 4 the functions are independent of the dimension of the convex body.*

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