Existence of quadrature surfaces for uniform density supported by a segment

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Abstract. Given are two strictly positive constants a and k. We show that if $a \geq 3.92k$ then there exists an open and bounded set Ω in $\mathbb{R}^2$ which contains strictly the line segment C ($C = [-1, 1] \times \{0\}$) such that the following overdetermined problem has a solution

$$-\Delta u = a\delta_C \text{ in } \Omega, \quad u = 0 \text{ and } -\frac{\partial u}{\partial \nu} = k \text{ on } \partial\Omega.$$

Here ν is the outward normal vector to $\partial\Omega$ and δ_C is the uniform density supported by C.

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1 Introduction and main theorem

Let μ be a positive measure with compact support K_μ and denote by C_μ the convex hull of K_μ. Consider the following free boundary problem:

$$(FB_\mu) \begin{cases} \text{Find a domain } \Omega \text{ of } \mathbb{R}^N \text{ which strictly contains } K_\mu \\ \text{and a function } u_\Omega \in H_0^1(\Omega) \text{ such that:} \\ \begin{cases} -\Delta u_\Omega = \mu \text{ in } \Omega \\ u_\Omega = 0 \quad \text{on } \partial\Omega \\ -\frac{\partial u_\Omega}{\partial \nu} = k \text{ on } \partial\Omega \text{ (overdetermined condition).} \end{cases} \end{cases}$$

where ν is the outward normal vector to $\partial\Omega$ and $k \in \mathbb{R}_+^*$.

This problem is known as the quadrature surfaces free boundary problem and arises in many areas of physics (free streamlines, jets, Hele-shaw flows, electromagnetic shaping, gravitational problems etc.) It has been intensively studied from different points of view, by several authors. For more details about the methods used for solving this problem see the introduction in [6].

In [2], the authors gave sufficient condition of existence for the problem (FB_μ) with $\mu \in L^2(\mathbb{R}^N)$ $(N \geq 2)$ and K_μ has a nonempty interior.

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This paper concerns the case where $N = 2$ and $\mu = a\delta_{[-1,1]\times\{0\}}$ $(a > 0)$.

By using the moving plane method [5], H. Shahgholian showed in [9] that if the problem (FB_μ) admits a solution (Ω, u_Ω) such that Ω is of class C^2 and $u_\Omega \in C^2(\overline{\Omega})$, then all the inward normals at the boundary $\partial\Omega$ of Ω meet C_μ. Since we relate the existence of a solution for Problem (FB_μ) to the existence of a minimum of some shape optimization problem, it is natural to resolve this one in a class of domains with this geometric normal property (see below).

In [1], the author studied bounded domains with the property he denoted by C-GNP (Geometric Normal Property w.r.t C). Namely, for a given compact convex set C, the bounded domain ω satisfies C-GNP if

1. $\omega \supset \mathrm{int}(C)$,
2. $\partial\omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, and
4. for every $x \in \partial\omega \setminus C$ the inward normal ray to ω (if exists) meets C.

Let D be a ball of $\mathbb{R}^2$ with large radius in order to contain all the sets we will use. Let $C = [-1, 1] \times \{0\}$ and set

$$\mathcal{O}_C = \{\omega \subset D \ : \ \omega \text{ satisfies } C - \text{GNP}\},$$

and

$$J(\omega) = -\frac{1}{2}\int_\omega |\nabla u_\omega(x)|^2 dx + \frac{k^2}{2}\int_\omega dx,$$

where u_ω is the solution of the following Dirichlet problem $P(\omega)$:

$$-\Delta u_\omega = a\delta_C \ \text{ in } \ \omega, \ \ u_\omega = 0 \ \text{ on } \ \partial\omega.$$

Remark 1.1. δ_C is a distribution belonging to $H^{-1}(\omega)$ and thus the solution u_ω of $P(\omega)$ is a priori only in $H_0^1(\omega)$. Nevertheless u_ω is harmonic (and thus it is C^∞) outside the line segment C and one can prove that it is continuous in ω.

Our aim here is to prove the following

Theorem 1.2. *1. If $a \geq 3.92k$ then there exists $\Omega \in \mathcal{O}_C$ which contains strictly C and such that $J(\Omega) = \min_{\omega\in\mathcal{O}_C} J(\omega)$ and*

$$\begin{cases} -\Delta u_\Omega = a\delta_C \ \ \text{in } \ \Omega \\ u_\Omega = 0 \quad \text{on } \partial\Omega \end{cases}$$

2. If Ω *is of class* C^2 *then*

$$-\frac{\partial u_\Omega}{\partial \nu} = k \quad on \quad \partial\Omega.$$

For the first point, we will use the shape optimization tool in order to get the minimum Ω of J then the Standard Maximum Principle and the Fourier expansion will enable us to have a sufficient condition that C is strictly contained in Ω. For the second point, the shape derivative together with the characterization of the C-GNP (see Proposition 2.8 below) will give the overdetermined condition.

Remark 1.3. Theorem 1.2 says that if the line segment in the complex plane is provided with a uniform density above a certain level, then there will exist a domain containing compactly the line segment such that the given measure on the line segment is equigravitational to the arc-length measure of the domain.

2 Preliminary results

Definition 2.1. Let K_1 and K_2 be two compact subsets of D. We call a Hausdorff distance of K_1 and K_2 (or briefly $d_H(K_1, K_2)$), the following positive number:

$$d_H(K_1, K_2) = \max\left[\rho(K_1, K_2), \rho(K_2, K_1)\right],$$

where $\rho(K_i, K_j) = \max_{x\in K_i} d(x, K_j)\ \ i,\ j = 1,\ 2$ and $d(x, K_j) = \min_{y\in K_j} |x - y|$.

Definition 2.2. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. Let K_n and K be their complements in $\overline{D}$. We say that the sequence ω_n converges in the Hausdorff sense, to ω (or briefly $\omega_n \xrightarrow{H} \omega$) if

$$\lim_{n\to+\infty} d_H(K_n, K) = 0.$$

Definition 2.3. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. We say that the sequence ω_n converges in the compact sense, to ω (or briefly $\omega_n \xrightarrow{K} \omega$) if

- every compact subset of ω is included in ω_n, for n *large enough*, and
- every compact subset of $\overline{\omega}^c$ is included in $\overline{\omega}_n^c$, for n *large enough.*

Definition 2.4. Let ω_n be a sequence of open subsets of D and ω be an open subset of D. We say that the sequence ω_n converges in the sense of characteristic functions, to ω (or briefly $\omega_n \xrightarrow{L} \omega$) if χ_{ω_n} converges to χ_ω in $L^p_{loc}(\mathbb{R}^N)$, $p \neq \infty$, (χ_ω is the characteristic function of ω).

Theorem 2.5. *If* $\omega_n \in \mathcal{O}_C$*, then there exists an open subset* $\omega \subset D$ *and a subsequence (still denoted by* ω_n*) such that*

1. $\omega_n \xrightarrow{H} \omega$

2. $\omega_n \xrightarrow{K} \omega$

3. *χ_{ω_n} converges to χ_ω in $L^1(D)$*

4. $\omega \in \mathcal{O}_C$

5. *u_n converges strongly in $H_0^1(D)$ to u_ω (u_n and u_ω are respectively the solutions of $P(\omega_n)$ and $P(\omega)$).*

Furthermore, the assertions (1), (2) and (3) are equivalent.

For the proof of this theorem, see Theorem 3.1 and Theorem 4.3 in [1]. For the equivalence between (1), (2) and (3), see Propositions 3.4, 3.5, 3.6, 3.7 and 3.8 in [1]. Notice that, in general, we do not have the equivalence between (1), (2) and (3) (see for instance [7]).

Definition 2.6. Let C be a convex set. We say that an open subset ω has the C-SP, if and only if

1. $\omega \supset \text{int}(C)$,
2. $\partial\omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, and
4. $\forall\, x \in \partial\omega \setminus C \;\; K_x \cap \omega = \varnothing$,where K_x is the closed cone defined by

$$\left\{ y \in \mathbb{R}^N \;:\; (y - x).(z - x) \leq 0,\ \forall\, z \in C \right\}.$$

Remark 2.7. K_x is the normal cone to the convex hull of C and $\{x\}$.

Proposition 2.8. *ω has the C-GNP if and only if ω satisfies the C-SP.*

For the proof of this proposition see Proposition 2.3 in [1].

Theorem 2.9. *Let L be a compact subset of $\mathbb{R}^N$. Let f_n be a sequence a functions defined on L. We assume that the f_n are of class C^3 and*

$$\left| \frac{\partial f_n}{\partial x_i} \right| \leq M, \; \left| \frac{\partial^2 f_n}{\partial x_i \partial x_j} \right| \leq M, \; \left| \frac{\partial^3 f_n}{\partial x_i \partial x_j \partial x_k} \right| \leq M,$$

where M is a strictly positive constant and is independent of n.

Define a sequence Ω_n, by $\Omega_n = \{x \in L \;:\; f_n(x) > 0\}$ and suppose there exists $\alpha > 0$ such that $|f_n(x)| + |\nabla f_n(x)| \geq \alpha$ for all x in L. If the Ω_n have the C-GNP, then there exists Ω of class C^2 and a subsequence (still denoted by Ω_n) such that Ω_n converges in the compact sense, to Ω.

For the proof of this theorem, see [2].

Remark 2.10. The aim of Theorem 2.9 is to give the C^2 regularity of the minimum Ω of J defined below. This in order to use the shape derivative and so to resolve Problem $(FB)_\mu$. The proof of this theorem uses the following lemma

Lemma 2.11. *Let L be a compact subset of $\mathbb{R}^N$. Let f_n be a sequence of functions defined as Theorem 2.9. Suppose that Ω is an open subset of L such that*

$$\begin{aligned}\Omega &= \{x \in L \,:\, h(x) > 0\} \text{ and} \\ \partial\Omega &= \{x \in L \,:\, h(x) = 0\},\end{aligned}$$

where h is a continuous function defined in L. If the f_n converge uniformly to h in L, then the Ω_n converge in the compact sense, to Ω.

3 Proof of Theorem 1.2

3.1 Ω contains strictly C

Using the variational formulation of the Dirichlet problem $P(\omega)$, we get

$$\int_\omega |\nabla u_\omega(x)|^2 dx = a \int_C u_\omega.$$

If u_D denotes the solution of the Dirichlet problem $P(D)$, by the maximum principle, $0 \le u_\omega \le u_D$ and so

$$J(\omega) = -\frac{a}{2}\int_C u_\omega + \frac{k^2}{2}\int_\omega dx \ge -\frac{a}{2}\int_D u_D,$$

and $\inf J$ exists. Let Ω_n be a minimizing sequence in $\mathcal{O}_C$. One can choose Ω_n as in Theorem 2.9 above and get the existence of a subsequence Ω_{n_k} and of Ω which is of class C^2 such that $\Omega_{n_k} \xrightarrow{K} \Omega$. Then, from Theorem 2.5, item 1 implies $\Omega_{n_k} \xrightarrow{H} \Omega$, item 4 gives $\Omega \in \mathcal{O}_C$ and by item 3 $\int_{\Omega_{n_k}} dx$ converges to $\int_\Omega dx$. Now if u_{n_k} and u_Ω are respectively the solutions of $P(\Omega_{n_k})$ and $P(\Omega)$ then item 3 together with item 5 of Theorem 2.5 implies that $\int_{\Omega_{n_k}} u_{n_k}$ converges to $\int_\Omega u_\Omega$ when k tends to infinity. Hence $J(\Omega) = \min\limits_{\omega \in \mathcal{O}_C} J(\omega)$.

Now suppose, by contradiction, that $\partial\Omega$ intersects C at a point c. As we will reason locally, we can suppose that c is in the origin. Let $\varepsilon > 0$, put $\Omega_\varepsilon = \Omega \cup B(0,\varepsilon)$ and $I_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$. Let u_{Ω_ε} be the solution of the Dirichlet problem $P(\Omega_\varepsilon)$. By the maximum principle we have $u_{\Omega_\varepsilon} > u_\Omega$ in Ω. Then, as $(C \cap \Omega) \setminus I_\varepsilon \subset \Omega$ and $u_\Omega = 0 \le u_{\Omega_\varepsilon}$ on $C \setminus \Omega$, we get

$$-\frac{a}{2}\int_{C\setminus I_\varepsilon} u_{\Omega_\varepsilon} < -\frac{a}{2}\int_{C\setminus I_\varepsilon} u_\Omega.$$

Therefore

$$J(\Omega_\varepsilon) - J(\Omega) \le \frac{k^2}{2}\left(\int_{\Omega_\varepsilon} dx - \int_\Omega dx\right) + \frac{a}{2}\int_{I_\varepsilon} u_\Omega - \frac{a}{2}\int_{I_\varepsilon} u_{\Omega_\varepsilon} \tag{3.1}$$

$$\le \frac{\pi k^2 \varepsilon^2}{2} + \frac{a}{2}\int_{I_\varepsilon} u_\Omega - \frac{a}{2}\int_{I_\varepsilon} u_{\Omega_\varepsilon}. \tag{3.2}$$

To get a contradiction of our assumption, we need to prove the two following lemmas.

Lemma 3.1.

$$\int_{I_\varepsilon} u_\Omega \leq (k+\varepsilon)\varepsilon^2. \tag{3.3}$$

Proof. Since $0 \in \partial\Omega \cap C$, the optimality condition gives: $-\frac{\partial u_\Omega}{\partial \nu}(0) \leq k$ (see Remark 3.3 below). Now, as $u_\Omega = v - \frac{a}{2}|y|$ where v is harmonic in Ω then u_Ω is of class C^1on the closed higher half-plane. Consequently, for all $\varepsilon > 0$ there is a neighborhood $\mathcal{V}_\varepsilon$ of 0 in the higher half-plane, such that

$$\forall\, x \in\ \mathcal{V}_\varepsilon :\ |\nabla u_\Omega(x)| \leq k+\varepsilon. \tag{3.4}$$

The Mean-Value Theorem applied to the line segments $((-h,0),(0,0))$ and $((h,0),(0,0))$ $(0 < h < \varepsilon)$, (3.4) implies

$$u_\Omega(-h,0) \leq (k+\varepsilon)h \text{ and } u_\Omega(h,0) \leq (k+\varepsilon)h.$$

Therefore

$$\int_{-\varepsilon}^{\varepsilon} u_\Omega(h,0)dh \leq (k+\varepsilon)\varepsilon^2$$

which gives the inequality (3.3) . □

Lemma 3.2.

$$\int_{I_\varepsilon} u_{\Omega_\varepsilon} > \frac{2a\varepsilon^2}{\pi}\left[-\frac{1}{2}+\frac{\pi^2}{8}\right]. \tag{3.5}$$

Proof. Let v_ε be the solution of the following Dirichlet problem

$$(P_\epsilon)\ \begin{cases} -\Delta v_\varepsilon = a\delta_{I_\varepsilon} \ \text{ in } \ B(0,\varepsilon) \\ v_\varepsilon = 0 \ \text{ on } \ \partial B(0,\varepsilon). \end{cases}$$

By the maximum principle, we have $u_{\Omega_\varepsilon} > v_\varepsilon$ in $B(0,\varepsilon)$ and thus

$$\int_{I_\varepsilon} u_{\Omega_\varepsilon} > \int_{I_\varepsilon} v_\varepsilon.$$

Let $w = -\frac{a}{2}|y|$ be the fundamental solution of $-\Delta w = a\delta_{I_\varepsilon}$. If $w_\varepsilon = v_\varepsilon - w$ then w_ε satisfies:

$$\begin{cases} \Delta w_\varepsilon = 0 \ \text{ in } \ B(0,\varepsilon) \\ \\ w_\varepsilon = \dfrac{a\varepsilon}{2}|\sin(\theta)| \ \text{ on } \ \partial B(0,\varepsilon). \end{cases}$$

Since $w_\varepsilon(r,0) = v_\varepsilon(r,0)$ for all $r \in\ [0,\varepsilon]$, then

$$\int_{I_\varepsilon} v_\varepsilon = \int_{I_\varepsilon} w_\varepsilon.$$

But

$$w_\varepsilon(r,\theta) = \sum_{n\geq 0} a_n r^n \cos(n\theta) + \sum_{n\geq 1} b_n r^n \sin(n\theta),$$

$$w_\varepsilon(\varepsilon,\theta) = \frac{a\varepsilon}{2}|\sin(\theta)|$$

and

$$|\sin(\theta)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{p\geq 1}\frac{1}{(2p)^2-1}\cos(2p\theta)$$

then

$$w_\varepsilon(r,\theta) = \frac{a}{\pi}\left(\varepsilon - 2\sum_{p\geq 1}\frac{\varepsilon^{1-2p}}{(2p)^2-1}r^{2p}\cos(2p\theta)\right).$$

Therefore

$$\begin{aligned}
\int_{I_\varepsilon} w_\varepsilon(r,0)\,dr &= \int_{-\varepsilon}^{\varepsilon}(a_0 + \sum_{p\geq 1} a_{2p}r^{2p})dr \\
&= 2a_0\varepsilon + \sum_{p\geq 1}\frac{2a_{2p}\varepsilon^{2p+1}}{2p+1} \\
&= \frac{2a\varepsilon^2}{\pi}\left[1 - 2\sum_{p\geq 1}\frac{1}{(2p+1)((2p)^2-1)}\right].
\end{aligned}$$

Now, since

$$\sum_{p\geq 1}\frac{1}{(2p+1)((2p)^2-1)} = \frac{3}{4} - \frac{\pi^2}{16}$$

then

$$\int_{I_\varepsilon} u_{\Omega_\varepsilon} > \int_{I_\varepsilon} w_\varepsilon(r,0)dr = \frac{2a\varepsilon^2}{\pi}\left[-\frac{1}{2} + \frac{\pi^2}{8}\right].$$

□

End of the proof of Theorem 1.2

Now thanks to (3.3) and (3.5), the inequality 3.2 becomes

$$J(\Omega_\varepsilon) - J(\Omega) < \frac{1}{2}\left(ak + \pi k^2 - \frac{\pi^2-4}{4\pi}a^2\right)\varepsilon^2 + \frac{a}{2}\varepsilon^3.$$

or again

$$J(\Omega_\varepsilon) - J(\Omega) < \frac{k^2}{2}\left(\frac{a}{k} + \pi - 0.46\frac{a^2}{k^2}\right)\varepsilon^2 + \frac{a}{2}\varepsilon^3.$$

If we put $t = \frac{a}{k}$, the sign of

$$P(t) = -0.46t^2 + t + \pi$$

is negative if $t \geq 3.92$.

It follows that if $a \geq 3.92k$ then $J(\Omega_\varepsilon) < J(\Omega)$ which contradicts the minimality of Ω.

3.2 The overdetermined condition: $-\frac{\partial u_\Omega}{\partial \nu} = k$ on $\partial\Omega$

Put $J(\Omega) = \frac{1}{2}\left[J_1(\Omega) + k^2 V(\Omega)\right]$ where

$$J_1(\Omega) = -\int_\Omega |\nabla u_\Omega(x)|^2 dx,$$

and

$$V(\Omega) = \int_\Omega dx.$$

Let us consider a deformation field $\theta \in C^2\left(\mathbb{R}^N; \mathbb{R}^N\right)$. Since Ω is of class C^2, then the classical Hadamard formula gives the derivative of J with respect to the displacement θ (or in the direction θ) (see [8, 10]).

$$dJ_1(\Omega; \theta) = -\int_{\partial\Omega} \left(-\frac{\partial u_\Omega}{\partial \nu}\right)^2 \theta \cdot \nu \, d\sigma - 2\int_\Omega \nabla u_\Omega \cdot \nabla u'(x) dx$$

where ν is the outward normal vector to $\partial\Omega$ and u' the derivative of u_Ω which is defined as the solution of the following problem:

$$\text{(3.6)} \qquad \begin{cases} -\Delta u' = 0 \text{ in } \Omega \\ u' = -\dfrac{\partial u_\Omega}{\partial \nu}\theta \cdot \nu \text{ on } \partial\Omega, \end{cases}$$

and

$$dV(\Omega; \theta) = \int_{\partial\Omega} \theta \cdot \nu \, d\sigma.$$

Then

$$\text{(3.7)} \quad dJ(\Omega; \theta) = \frac{1}{2}\left[\int_{\partial\Omega} k^2 \theta \cdot \nu \, d\sigma - \int_{\partial\Omega} \left(-\frac{\partial u_\Omega}{\partial \nu}\right)^2 \theta \cdot \nu \, d\sigma\right] - \int_\Omega \nabla u_\Omega \cdot \nabla u'(x) dx.$$

Using the Green formula,

$$\begin{cases} dJ(\Omega; \theta) = \frac{1}{2}\left[\int_{\partial\Omega} k^2 \theta \cdot \nu \, d\sigma - \int_{\partial\Omega} \left(-\frac{\partial u_\Omega}{\partial \nu}\right)^2 \theta \cdot \nu \, d\sigma\right] \\ \qquad + \int_\Omega u_\Omega \Delta u'(x) dx - \int_{\partial\Omega} u_\Omega \dfrac{\partial u'}{\partial \nu} \theta \cdot \nu \, d\sigma. \end{cases}$$

According to (3.6) and (3.7),

$$dJ(\Omega; \theta) = \frac{1}{2}\left[\int_{\partial\Omega} k^2 \theta \cdot \nu \, d\sigma - \int_{\partial\Omega} \left(-\frac{\partial u_\Omega}{\partial \nu}\right)^2 \theta \cdot \nu \, d\sigma\right].$$

Now since Ω is the minimum of J, then $dJ(\Omega; \theta) \geq 0$ for every admissible displacement θ. Therefore

$$\text{(3.8)} \qquad \int_{\partial\Omega} \left(k^2 - \left(-\frac{\partial u_\Omega}{\partial \nu}\right)^2\right) \theta \cdot \nu \, d\sigma \geq 0 \text{ for every admissible displacement } \theta.$$

We mean by admissible displacement the one which allows us to keep the C-GNP or the C-SP (according to Proposition 2.8 above). Since Ω has the C-GNP, it satisfies the C-SP. This together with the fact that C is strictly contained in Ω implies

$$\forall\, x \in \partial\Omega \;\; K_x \cap \Omega = \varnothing\,.$$

For t sufficiently small, let $\Omega_t = \Omega + t\theta\,(\Omega)$ be the deformation of Ω in the direction θ. Let $x_t \in \partial\Omega_t$. There exists $x \in \partial\Omega$ s.t $x_t = x + t\theta(x)$. Using the definition of K_{x_t} and the equality above, it is obvious to get (for t small enough and for every displacement θ) :

$$\forall\, x_t \in \partial\Omega_t \;\; K_{x_t} \cap \Omega_t = \varnothing\,,$$

which means that Ω_t satisfies the C-SP (and so the C-GNP) for every displacement θ when t is sufficiently small. Then, using θ and $-\theta$, and the fact that the set of the functions $\theta \cdot \nu$ is dense in $L^2(\partial\Omega)$, we deduce

$$-\frac{\partial u_\Omega}{\partial \nu} = k \text{ on } \partial\Omega.$$

Remark 3.3. In the case where $\partial\Omega \cap C \neq \varnothing$, if there exists $x \in \partial\Omega \cap C$ such that $\theta\,(x) \cdot \nu\,(x) \leq 0$ then the inward normal at x_t doesn't intersect the line segment C. So to keep the C-GNP the displacements θ must satisfy $\theta\,(x) \cdot \nu\,(x) \geq 0$ for all $x \in \partial\Omega \cap C$. So (3.8) implies

$$-\frac{\partial u_\Omega}{\partial \nu}\,(x) \leq k \;\; \forall\, x \in \partial\Omega \cap C.$$

4 Final remarks

Remark 4.1. Using the notion of quadrature domains, B. Gustafsson and H. Shahgholian showed in [6] that in the case where $\mu = a\delta_{[-1,1]\times\{0\}}$ the problem (FB_μ) admits a solution if $a \geq 24\pi k$.

Remark 4.2. It is not hard to see that in the case where $\mu = a\delta_{[-1,1]\times\{0\}}$, if (Ω, u_Ω) is a regular solution of the problem (FB_μ) then $a > 2k$. **Question: Is the converse true?** The answer seems to be *no*.

Remark 4.3. Set

$$\mathcal{O}_P = \{\omega \in \mathcal{O}_C \text{ and } |\partial\omega| \leq cst\}\,.$$

where $|\partial\omega|$ denotes the perimeter of ω. Using the same arguments as above, one can prove that:

1. If $a \geq 3.92k$ then there exists $\Omega \in \mathcal{O}_P$ which contains strictly C and such that $J\,(\Omega) = \min\limits_{\omega \in \mathcal{O}_P} J\,(\omega)$ and

$$\begin{cases} -\Delta u_\Omega = a\delta_C & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega \end{cases}$$

2. If Ω is of class C^2 then there exists a Lagrangian multiplier $\lambda(\Omega)$ s.t

$$-\frac{\partial u_\Omega}{\partial \nu} = \sqrt{\lambda(\Omega) H_{\partial\Omega} + k^2} \quad \text{on} \quad \partial\Omega.$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$.

Remark 4.4. In [3], the authors gave sufficient condition of existence for the following free boundary problem

$$\begin{cases} -\Delta u_\Omega = \mu \;\text{ in }\; \Omega \\ u_\Omega = 0 \quad \text{on } \partial\Omega \\ -\frac{\partial u_\Omega}{\partial \nu} = \sqrt{\lambda H_{\partial\Omega} + k^2} \text{ on } \;\partial\Omega \end{cases}$$

where **Supp**μ has a nonempty interior and λ and k are two positive constants.

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