On the Blaschke invariants of generalized time-like ruled surfaces

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Abstract. In this work, we give the relations between Blaschke invariants and principal distribution parameters of $(k+1)-$dimensional time-like ruled surfaces in $n-$dimensional Minkowski space $\mathbf{R}_1^n$. In addition to this we have obtained statements for $2-$dimensional space-like and time-like ruled surfaces generated by a unit vector $e(t)$ of the generator space $E_k(t)$. Finally some examples for all the cases are also given.

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§1. Introduction

In literature there are many studies related to ruled surfaces and their invariants (distribution parameters, Blaschke invariants, apex angles, etc.) in $n-$dimensional Euclidean space E^n and 3−dimensional Euclidean space E^3, [3, 4, 5, 11]. It is well known that the geometry of ruled surfaces is very important in the study of kinematics or spatial mechanisms in E^3, [6, 8].

Lorentz metrics in $n-$dimensional Minkowski space $\mathbf{R}_1^n$ is indefinite. In the theory of relativity, geometry of indefinite metric is very crucial. Hence, the theory of ruled surfaces in Minkowski space $\mathbf{R}_1^3$, which has the metric, $ds^2 = dx^2 + dy^2 - dz^2$ attracted much attention. A series of papers are devoted to the construction of the ruled surfaces, [14, 15, 16].

The situation is much more complicated than the Euclidean case, since the ruled surfaces may have a definite metric (spacelike surfaces), Lorentz metric (time-like surfaces) or mixed metric. Recently, the time-like or space-like ruled surfaces in $\mathbf{R}_1^3$ and $\mathbf{R}_1^n$ have been studied systematically, [1, 2, 10, 12, 13, 14, 15, 16].

This paper is organized as follows. In the first part basic concepts have been given in Minkowski space, $\mathbf{R}_1^n$. In the next part, $(k+1)-$dimensional time-like ruled surfaces, their asymptotic bundles and tangential bundles are defined in $\mathbf{R}_1^n$. Furthermore, the knowledge about the edge ruled surfaces and central ruled surface of $(k+1)-$dimensional time-like ruled surface are given.

Finally, the relations between Blaschke invariants of $(k+1)-$dimensional time-like ruled surface with distribution parameter and principal distribution parameter have also been expressed. After all, Blaschke invariants of $2-$dimensional time-like and

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space-like ruled surfaces in $\mathbf{R}_1^n$ have been evaluated and examples related to these ruled surfaces have been given.

§2. Preliminaries

We note, first of all, that the notation and fundamental formulas have been used in this study as [10]. The Minkowski space $\mathbf{R}_1^n$ is the vector space $\mathbf{R}^n$ provided that the Lorentzian inner product g is given by

$$g = dx_1^2 + dx_2^2 + ... + dx_{n-1}^2 - dx_n^2$$

where $(x_1, x_2, ..., x_n)$ is rectangular coordinate system of $\mathbf{R}_1^n$. Since g is indefinite metric, recall that a vector $v \in \mathbf{R}_1^n$ can have one of three Lorentzian casual characters: it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null $g(v, v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s) \subset \mathbf{R}_1^n$ can locally be space-like, time-like or null (light-like), if all of its velocity vectors $\dot{\alpha}(s)$ are respectively space-like, time-like or null (light-like). The norm of a vector $v \in \mathbf{R}_1^n$ is defined as

$$\|v\| = \sqrt{|g(v, v)|}.$$

Let W be a subspace of $\mathbf{R}_1^n$ and denote g_W as reduced metric in subspace W of $\mathbf{R}_1^n$. A subspace W of $\mathbf{R}_1^n$ can be space-like, time-like or null (light-like) if g_W is positive definite, g_W is non-degenerate with index 1 or g_W is degenerate, respectively.
Let the set of all time-like vectors in $\mathbf{R}_1^n$ be Γ. For $u \in \Gamma$, we call

$$C(u) = \{ v \in \Gamma | \, g(v, u) < 0\}$$

as time-conic of Minkowski space $\mathbf{R}_1^n$ including vector u.

Lemma 1. *Let v and w be two time-like vectors in Minkowski space $\mathbf{R}_1^n$. In this case there exists the following inequality*

$$|g(v, w)| \geq \|v\| \, \|w\| .$$

In this inequality if one wishes the equality condition, then it is necessary for v and w be linear dependent.
If time-like vectors v and w stay inside the same time conic, then there is a single number of $\theta \geq 0$ such that

$$g(v, w) = -\|v\| \, \|w\| \cosh \theta$$

where the number θ is called an angle between the time-like vectors, [10].
Let v and w be two space-like vectors in Minkowski space $\mathbf{R}_1^n$. Then, we write the following inequality

$$|g(v, w)| \geq \|v\| \, \|w\| .$$

If v and w are two space-like vectors in Minkowski space, then there is a single number of $0 \leq \theta \leq \pi$ such that

$$g(v, w) = \|v\| \, \|w\| \cos \theta$$

where the number θ is called the angle between space-like vectors v and w, [10]. Considering v and w to be space-like and time-like vector, respectively, in Minkowski space $\mathbf{R}_1^n$ there is a number θ such that

$$g(v,w) = \|v\| \, \|w\| \sinh\theta$$

where the number θ is called the angle between space-like vector v and time-like vector w, [9].

§3. Time-like ruled surfaces

Let $\{e_1(t), ..., e_k(t)\}$ be an orthonormal vector field, which is defined at each point $\alpha(t)$ of a space-like curve of a $n-$dimensional Minkowski space, $\mathbf{R}^n$. This system spanning at the point $\alpha(t) \in \mathbf{R}_1^n$ a $k-$dimensional subspace is denoted by $E_k(t)$ and is given by $E_k(t) = Sp\{e_1(t), ..., e_k(t)\}$. If the subspace $E_k(t)$ moves along the curve α, we obtain a $(k+1)-$dimensional surface in $\mathbf{R}^n$. This surface is called a $(k+1)-$dimensional time-like ruled surface of the $n-$dimensional Minkowski space, $\mathbf{R}^n$ and is denoted by M.
The subspace $E_k(t)$ and space-like curve α are called the generating space and the base curve, respectively. A parametrization of the ruled surface is the following:

$$\phi(t, u_1, ...u_k) = \alpha(t) + \sum_{i=1}^{k} u_i e_i(t). \tag{3.1}$$

Throughout the paper we assume that the system $\left\{\dot{\alpha}(t) + \sum\limits_{i=1}^{k} u_i \dot{e}_i(t), e_1(t), ..., e_k(t)\right\}$ is linear independent and is time-like subspace, [1]. We call

$$Sp\{e_1(t), ..., e_k(t), \dot{e}_1(t), ...\dot{e}_k(t)\} \tag{3.2}$$

the asymptotic bundle of M with respect to $E_k(t)$ and denote it by $A(t)$. We have $\dim A(t) = k+m$, $0 \leq m \leq k$. There exists an orthonormal base of $A(t)$ that we denote as $\{e_1(t), ...e_k(t), a_{k+1}(t), ...a_{k+m}(t)\}$. It is clear that the asymptotic bundle is time-like subspace. The space

$$Sp\{e_1(t), ..., e_k(t), \dot{e}_1(t), ..., \dot{e}_k(t), \dot{\alpha}(t)\} \tag{3.3}$$

includes the union of all the tangent spaces of $E_k(t)$ at a point p. This space is denoted by $T(t)$ and called the tangential bundle of M in $E_k(t)$. It can be easily seen that

$$k+m \leq \dim T(t) \leq k+m+1 \quad , \quad 0 \leq m \leq k.$$

In what follow we examine separately two cases. Let $\dim T(t) = k+m$, then $\{e_1(t), ...e_k(t), a_{k+1}(t), ...a_{k+m}(t)\}$ is an orthonormal bases of the asymptotic bundle $A(t)$ and the tangential bundle $T(t)$. Let $\dim T(t) = k+m+1$, then $\{e_1(t), ...e_k(t), a_{k+1}(t), ...a_{k+m}(t), a$ is an orthonormal bases of $T(t)$. The tangential bundle $T(t)$ is in both cases a time-like subspace, [1].
If $\dim T(t) = k+m$, then $(k+1)-$dimensional time-like ruled surface M has $(k-m)-$dimensional

subspace and this subspace is called edge space of M and denoted as $K_{k-m}(t) \subset E_k(t)$. Edge space $K_{k-m}(t) \subset E_k(t)$ is either time-like or space-like subspace. If we take edge space $K_{k-m}(t)$ to be generated space and base curve α of M to be base curve, then there will be $(k-m+1)$ −dimensional ruled surface contained by M. This surface is called edge ruled surface. If $K_{k-m}(t)$ is space-like (time-like), then edge ruled surface becomes space-like (time-like) ruled surface, [1].

If $\dim T(t) = k+m+1$, then $(k+1)$ −dimensional time-like ruled surface has a $(k-m)$ −dimensional subspace called central space of M and denoted as $Z_{k-m}(t) \subset E_k(t)$. This space is either time-like or space-like subspace. Similarly, if we take base curve of M to be the base curve and $Z_{k-m}(t)$ to be the generating space, we get a $(k-m+1)$ −dimensional ruled surface contained by M in $\mathbf{R}_1^n$ and this is called central ruled surface and denoted by Ω. If $Z_{k-m}(t)$ is space-like (time-like), then central ruled surface Ω is space-like (time-like) surface, [1].

Let us complete the orthonormal basis $\{e_1(t), ...e_k(t), a_{k+1}(t), ...a_{k+m}(t), a_{k+m+1}(t)\}$ of tangential bundle $T(t)$ to the orthonormal basis
$\{e_1(t), ...e_k(t), a_{k+1}(t), ...a_{k+m}(t), a_{k+m+1}(t), a_{k+m+2}(t), ..., a_n(t)\}$
of $\mathbf{R}_1^n$. For these basis vectors we write the following derivative equations, [1];

(3.4)

$$\dot{e}_\sigma(t) = \sum_{v=1}^{k} \alpha_{\sigma\nu} e_\nu(t) + \kappa_\sigma a_{k+\sigma}(t), \qquad 1 \le \sigma \le m$$

$$\dot{e}_{m+p}(t) = \sum_{v=1}^{k} \alpha_{(m+p)\nu} e_\nu(t), \qquad 1 \le p \le k-m$$

$$(\kappa_\sigma = \|\dot{e}_\sigma\| \ge 0 \quad , \quad \varepsilon_\nu \alpha_{\sigma\nu} = -\alpha_{\nu\sigma} \quad , \quad \varepsilon_\nu = \langle e_\nu, e_\nu \rangle = \mp 1)$$

$$\dot{a}_{k+\sigma}(t) = -\varepsilon_\sigma \kappa_\sigma e_\sigma(t) + \sum_{l=1}^{m} \tau_{\sigma l} a_{k+l}(t) + \omega_\sigma a_{k+m+1}(t) + \sum_{\lambda=2}^{n-k-m} \gamma_{\sigma\lambda} a_{k+m+\lambda}(t), \quad 1 \le \sigma \le m$$

$$\dot{a}_{k+m+1}(t) = -\sum_{l=1}^{m} \omega_l a_{k+l}(t) - \sum_{\lambda=2}^{n-k-m} \beta_\lambda a_{k+m+\lambda}(t)$$

$$\dot{a}_{k+m+s}(t) = -\sum_{l=1}^{m} \omega_{sl} a_{k+l}(t) + \beta_s a_{k+m+1}(t) + \sum_{\lambda=2}^{n-k-m} \beta_{s\lambda} a_{k+m+\lambda}(t), \quad 2 \le s \le n-k-m$$

$$(\tau_{\sigma l} = -\tau_{l\sigma} \quad , \quad \beta_{s\lambda} = -\beta_{\lambda s} \quad , \quad \omega_{sl} = -\gamma_{\sigma\lambda})$$

Let subspace $F_m(t) = Sp\{e_1(t), ...e_m(t)\}$ be totally orthogonal to generating space $Z_{k-m}(t)$ of Ω and orthogonal trajectories of central ruled surface Ω be r. If generating space $F_m(t)$ moves along base curve r, it produces a $(m+1)$ −dimensional ruled surface. This surface is known as principal ruled surface and denoted by Λ, [1].

If central ruled surface Ω is time-like, then direction vector $F(t)$ along space-like curve α produces 2−dimensional space-like ruled surfaces of number m called principal ray surfaces since each direction vector $F(t) = Sp\{e_i(t)\}$, $1 \le i \le m$ is space-like, [1].

If central ruled surface Ω is space-like, since one direction vector of $F(t) = Sp\{e_i(t)\}$, $1 \le i \le m$ is time-like and $(m-1)$ direction vectors are space-like. Direction vector $F(t)$ along space-like curve α will produce one time-like and $(m-1)$ space-like principal ray ruled surfaces. These principal ray ruled surfaces are parametrized as follows:

$$\phi_i(t, x) = \alpha(t) + x e_i(t).$$

If $(k+1)$ −dimensional time-like ruled surface M is cylindrical (i.e., $m = 0$), then there is no principal ray ruled surface of M. A base curve α of $(k+1)$ −dimensional ruled surface M is a base curve of edge or central ruled surface $\Omega \subset M$, too iff its tangent vector has the form

$$\dot{\alpha}(t) = \sum_{i=1}^{k} \zeta_i e_i + \eta_{m+1} a_{k+m+1} \tag{3.5}$$

where $\eta_{m+1} \neq 0$, a_{k+m+1} is a unit vector well defined up to the sign with the property that $\{e_1, ..., e_k, a_{k+1}, ..., a_{k+m}, a_{k+m+1}\}$ is an orthonormal base of the tangential bundle of M. One shows: $\eta_{m+1} = 0$, in $t \in J$ iff generator $E_k(t) \subset M$ contains the edge space $K_{k-m}(t)$, [1].
If $\eta_{m+1} \neq 0$, we call $m-$magnitudes

$$P_i = \frac{\eta_{m+1}}{\kappa_i} \quad , \quad 1 \leq i \leq m \tag{3.6}$$

the principal parameter of distribution, [1]. Moreover, in [1] the parameter of distribution of a generalized ruled surface M is given by

$$P = \sqrt[m]{|P_1 ... P_m|} \tag{3.7}$$

and the total parameter of distribution of M is defined to be

$$D = \prod_{i=1}^{m} P_i. \tag{3.8}$$

For two-dimensional ruled surface M for which the base curve is space-like curve α and its generating space is $E(t) = Sp\{e(t)\}$ in $\mathbf{R}_1^n$, the magnitude of

$$b = \frac{\zeta}{\kappa} \tag{3.9}$$

is called Blaschke invariant of M, [5].
Let M be a $(k+1)-$dimensional time-like ruled surface in $\mathbf{R}_1^n$. If dimension of the asymptotic bundle $A(t)$ of M is $(k+m)$, then the magnitudes

$$b_i = \frac{\zeta_i}{\kappa_i} \quad , \quad 1 \leq i \leq m \tag{3.10}$$

are called the principal Blaschke invariants of M and

$$B = \sqrt[m]{|b_1 ... b_m|} \tag{3.11}$$

is called the Blaschke invariant of M, [7].

§4. The Blaschke invariants of ruled surfaces in $\mathbf{R}_1^n$

In this section, the Blaschke invariants of two-dimensional and $(k+1)-$dimensional time-like ruled surface in $n-$dimensional Minkowski space $\mathbf{R}_1^n$ and relations between these Blaschke invariants and distribution parameters are expressed.

Theorem 1. *Let M be a $(k+1)-$dimensional time-like ruled surfaces with central ruled surfaces Ω. The relation between the Blaschke invariants and principal parameter of M is*

$$B = \frac{1}{\eta_{m+1}} \sqrt[m]{\left| \prod_{i=1}^{m} \zeta_i P_i \right|}. \tag{4.12}$$

Proof. From equation (3.6) we know that the principal distribution parameter of M is

$$P_i = \frac{\eta_{m+1}}{\kappa_i} \quad , \quad 1 \leq i \leq m.$$

From this last equation, we can write

$$\kappa_i = \frac{\eta_{m+1}}{P_i} \quad , \quad 1 \leq i \leq m$$

and substituting this equation into equation (3.10) we reach

$$b_i = \frac{1}{\eta_{m+1}} \zeta_i P_i \quad , \quad 1 \leq i \leq m.$$

The last equation with equation (3.11) completes the proof. □
This theorem with equation (3.7) and (3.8) give us the following results.

Result 1. *In $\mathbf{R}_1^n$, the relation between distribution parameter and Blaschke invariants of M is*

$$B = \frac{1}{\eta_{m+1}} \sqrt[m]{\left| \prod_{i=1}^{m} \zeta_i \right| P}. \tag{4.13}$$

Result 2. *In $\mathbf{R}_1^n$, the relation between total distribution parameter and Blaschke invariants of M is*

$$B = \frac{1}{\eta_{m+1}} \sqrt[m]{\left| D \prod_{i=1}^{m} \zeta_i \right|}. \tag{4.14}$$

In $\mathbf{R}_1^n$, let M be $(k+1)-$dimensional time-like ruled surface with dimensional central ruled surface Ω. If $m = k$, then the central ruled surface Ω of M will be degenerate in the sitriction line of M. this case, $1-$dimensional generating space $Sp\{e(t)\} = E(t)$, which is inside the generating space $E_k(t)$ of M produces $2-$dimensional ruled surface ψ. The generating space $E(t) = Sp\{e(t)\} \subset E_k(t)$ (unitary direction vector) is either time-like or space-like. Now we consider these two cases separately.
First, we suppose that the subspace $E(t) = Sp\{e(t)\}$ is time-like subspace of $E_k(t)$. Then, for unit time-like vector $e(t)$ we can write

$$e(t) \in Sp\{e_1(t), ..., e_k(t)\} \quad , \quad \|e(t)\| = 1$$

and

(4.15)
$$e(t)=\sum_{\nu=1}^{k-1}\sinh\theta_\nu e_\nu+\cosh\theta_k e_k \quad , \quad \sum_{\nu=1}^{k-1}\sinh^2\theta_\nu-\cosh^2\theta_k=1 \quad , \quad \theta_\nu,\theta_k=\text{constant}$$

where the angles between unit time-like vector $e(t)$ and unit vectors $e_1(t),...,e_k(t)$ are $\theta_1,\theta_2,...\theta_k$, respectively. Since $E_k(t)=Sp\{e_1(t),...,e_k(t)\}$ is time-like subspace we adopt that $e_1(t),...,e_{k-1}(t)$ vectors as space-like and $e_k(t)$ to be a time-like vector. Therefore we can give the following theorem.

Theorem 2. *Let ψ ($\psi\subset M$) be $2-$dimensional time-like ruled surface in $n-$dimensional Minkowski space $\mathbf{R}_1^n$. The Blaschke invariant of $2-$dimensional time-like ruled surface ψ is*

$$b=\frac{\sum\limits_{\nu=1}^{k-1}\zeta_\nu\sinh\theta_\nu-\zeta_k\cosh\theta_k}{\sqrt{\left|\begin{array}{l}\sum\limits_{\mu=1}^{k-1}\left[\left(\sum\limits_{\nu=1}^{k-1}\alpha_{\nu\mu}\sinh\theta_\nu+\alpha_{k\mu}\cosh\theta_k\right)^2+(\kappa_\mu\sinh\theta_\mu)^2\right]+(\kappa_k\cosh\theta_k)^2\\ -\left(\sum\limits_{\nu=1}^{k-1}\alpha_{\nu k}\sinh\theta_\nu+\alpha_{kk}\cosh\theta_k\right)^2\end{array}\right|}}.$$

Proof. Considering equations (3.5) and (4.15) gives us

$$\zeta=\langle\dot\alpha(t),e(t)\rangle=\left\langle\sum_{\mu=1}^{k}\zeta_\mu e_\mu(t)+\eta_{m+1}a_{k+m+1}(t),\sum_{\nu=1}^{k-1}\sinh\theta_\nu e_\nu(t)+\cosh\theta_k e_k(t)\right\rangle$$

$$\zeta=\sum_{\nu=1}^{k-1}\zeta_\nu\sinh\theta_\nu-\zeta_k\cosh\theta_k. \tag{4.16}$$

From equation (4.15) we get

$$\dot e(t)=\sum_{\nu=1}^{k-1}\sinh\theta_\nu\dot e_\nu(t)+\cosh\theta_k\dot e_k(t).$$

Substituting equation (3.4) into the last equation above we reach

$$\dot e(t)=\sum_{\nu=1}^{k-1}\sinh\theta_\nu\left(\sum_{\mu=1}^{k}\alpha_{\nu\mu}e_\mu(t)+\kappa_\nu a_{k+\nu}(t)\right)+\cosh\theta_k\left(\sum_{\mu=1}^{k}\alpha_{k\mu}e_\mu(t)+\kappa_k a_{2k}(t)\right)$$

(4.17)
$$\dot e=\sum_{\mu=1}^{k}\left(\sum_{\nu=1}^{k-1}\alpha_{\nu\mu}\sinh\theta_\nu+\alpha_{k\mu}\cosh\theta_k\right)e_\mu(t)+\sum_{\nu=1}^{k-1}\kappa_\nu\sinh\theta_\nu a_{k+\nu}(t)+\kappa_k\cosh\theta_k a_{2k}(t).$$

Thus we find

(4.18)

$$\kappa = \sqrt{\left| \begin{array}{l} \sum\limits_{\mu=1}^{k-1}\left[\left(\sum\limits_{\nu=1}^{k-1}\alpha_{\nu\mu}\sinh\theta_\nu + \alpha_{k\mu}\cosh\theta_k\right)^2 + (\kappa_\mu\sinh\theta_\mu)^2\right] + (\kappa_k\cosh\theta_k)^2 \\ -\left(\sum\limits_{\nu=1}^{k-1}\alpha_{\nu k}\sinh\theta_\nu + \alpha_{kk}\cosh\theta_k\right)^2 \end{array} \right|}.$$

Substituting equations (4.16) and (4.18) into equation (3.10) we find

$$b = \frac{\sum\limits_{\nu=1}^{k-1}\zeta_\nu\sinh\theta_\nu - \zeta_k\cosh\theta_k}{\sqrt{\left| \begin{array}{l} \sum\limits_{\mu=1}^{k-1}\left[\left(\sum\limits_{\nu=1}^{k-1}\alpha_{\nu\mu}\sinh\theta_\nu + \alpha_{k\mu}\cosh\theta_k\right)^2 + (\kappa_\mu\sinh\theta_\mu)^2\right] + (\kappa_k\cosh\theta_k)^2 \\ -\left(\sum\limits_{\nu=1}^{k-1}\alpha_{\nu k}\sinh\theta_\nu + \alpha_{kk}\cosh\theta_k\right)^2 \end{array} \right|}}.$$

□

Now we suppose that $E(t) = Sp\{e(t)\}$ is space-like. Therefore, for unit space-like vector $e(t)$ it follows that

$$e(t) \in Sp\{e_1(t), ..., e_k(t)\} \quad , \quad \|e(t)\| = 1$$

and

(4.19)

$$e(t) = \sum_{\nu=1}^{k-1}\cos\theta_\nu e_\nu - \sinh\theta_k e_k \quad , \quad \sum_{\nu=1}^{k-1}\cos^2\theta_\nu - \sinh^2\theta_k = 1 \quad , \quad \theta_\nu, \theta_k = \text{constant}$$

where, as we did in the first assumption, the angles between unit vectors $e_1(t), ..., e_k(t)$ and unit space-like vector $e(t)$ are $\theta_1, \theta_2, ...\theta_k$, respectively. In the subspace $E_k(t) = Sp\{e_1(t), ..., e_k(t)\}$, $e_k(t)$ will be time-like whereas others are space-like vectors. Thus we can give the following theorem.

Theorem 3. *Let ψ ($\psi \subset M$) be $2-$dimensional space-like ruled surface in $n-$dimensional Minkowski space $\mathbf{R}_1^n$. The Blaschke invariant of $2-$dimensional space-like ruled surface ψ is*

$$b = \frac{\sum\limits_{\nu=1}^{k-1}\zeta_\nu\cos\theta_\nu + \zeta_k\sinh\theta_k}{\sqrt{\left| \begin{array}{l} \sum\limits_{\mu=1}^{k-1}\left[\left(\sum\limits_{\nu=1}^{k-1}\alpha_{\nu\mu}\cos\theta_\nu - \alpha_{k\mu}\sinh\theta_k\right)^2 + (\kappa_\mu\cos\theta_\mu)^2\right] + (\kappa_k\sinh\theta_k)^2 \\ -\left(\sum\limits_{\nu=1}^{k-1}\alpha_{\nu k}\cos\theta_\nu - \alpha_{kk}\sinh\theta_k\right)^2 \end{array} \right|}}.$$

Proof. Considering equations (3.5) and (4.19) we find

$$\zeta = \langle \dot{\alpha}(t), e(t) \rangle = \left\langle \sum_{\mu=1}^{k} \zeta_\mu e_\mu(t) + \eta_{m+1} a_{k+m+1}(t), \sum_{\nu=1}^{k-1} \cos\theta_\nu e_\nu(t) - \sinh\theta_k e_k(t) \right\rangle$$

$$\zeta = \sum_{\nu=1}^{k-1} \zeta_\nu \cos\theta_\nu + \zeta_k \sinh\theta_k. \tag{4.20}$$

From equation (4.19) we get

$$\dot{e}(t) = \sum_{\nu=1}^{k-1} \cos\theta_\nu \dot{e}_\nu(t) - \sinh\theta_k \dot{e}_k(t).$$

Substituting equation (3.4) into the last equation gives us

$$\dot{e}(t) = \sum_{\nu=1}^{k-1} \cos\theta_\nu \left(\sum_{\mu=1}^{k} \alpha_{\nu\mu} e_\mu(t) + \kappa_\nu a_{k+\nu}(t) \right) - \sinh\theta_k \left(\sum_{\mu=1}^{k} \alpha_{k\mu} e_\mu(t) + \kappa_k a_{2k}(t) \right)$$

$$\dot{e}(t) = \sum_{\mu=1}^{k} \left(\sum_{\nu=1}^{k-1} \alpha_{\nu\mu} \cos\theta_\nu - \alpha_{k\mu} \sinh\theta_k \right) e_\mu(t) + \sum_{\nu=1}^{k-1} \kappa_\nu \cos\theta_\nu a_{k+\nu}(t) - \kappa_k \sinh\theta_k a_{2k}(t). \tag{4.21}$$

Therefore, from equations (3.4) and (4.21) we find

$$\kappa = \sqrt{\left| \begin{array}{l} \sum\limits_{\mu=1}^{k-1} \left[\left(\sum\limits_{\nu=1}^{k-1} \alpha_{\nu\mu} \cos\theta_\nu - \alpha_{k\mu} \sinh\theta_k \right)^2 + \left(\kappa_\mu \cos\theta_\mu \right)^2 \right] + \left(\kappa_k \sinh\theta_k \right)^2 \\ - \left(\sum\limits_{\nu=1}^{k-1} \alpha_{\nu k} \cos\theta_\nu - \alpha_{kk} \sinh\theta_k \right)^2 \end{array} \right|}. \tag{4.22}$$

Substituting (4.20) and (4.22) into equation (3.10) completes the proof. □

Example 1. *In* $3-$*dimensional Minkowski space* $\mathbf{R}_1^3$ *let us define*

$$\Phi(s,v) = \left(\sqrt{2}\cos s - 2\sqrt{2} v \sin s, \sqrt{2}\sin s + 2\sqrt{2} v \cos s, s + 3v \right)$$

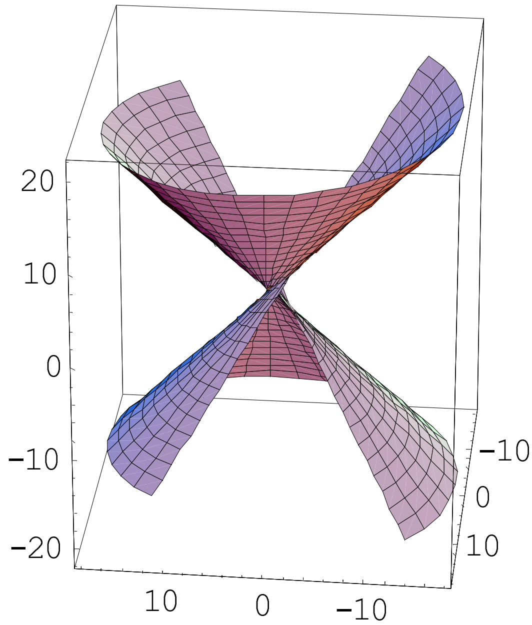
to be the space-like base curve of $2-$*dimensional time-like ruled surface given by*

$$\alpha(s) = \left(\sqrt{2}\cos s, \sqrt{2}\sin s, s \right)$$

and generating vector to be

$$e(s) = \left(-2\sqrt{2}\sin s, 2\sqrt{2}\cos s, 3 \right)$$

where the time-like subspace $E(t) = Sp\{e(t)\}$ is generating space of Φ, (see Figure).



Velocity vector of base curve α is

$$\dot{\alpha}(s) = \left(-\sqrt{2}\sin s, \sqrt{2}\cos s, 1\right).$$

Let us complete $E(t) = Sp\{e(t)\}$ to orthonormal base $\{e, a_2, a_3\}$ of $\mathbf{R}_1^3$ as

$$a_2(s) = (-\cos s, -\sin s, 0),$$
$$a_3(s) = \left(-3\sin s, 3\cos s, 2\sqrt{2}\right).$$

Derivative equations of these base vectors are found to be

$$\dot{e}(s) = 2\sqrt{2}a_2(s),$$
$$\dot{a}_2(s) = 2\sqrt{2}e(s) - 3a_3(s),$$
$$\dot{a}_3(s) = 3a_2(s).$$

Therefore, the Blaschke invariant of $2-$dimensional time-like ruled surface Φ and the distribution parameter are evaluated to be

$$b = \tfrac{\zeta}{\kappa} = -\tfrac{\sqrt{2}}{4},$$
$$P = \tfrac{\eta}{\kappa} = \tfrac{1}{2},$$

respectively. Thus, there is a relation between Blaschke invariant and distribution parameter of $2-$dimensional time-like ruled surface Φ as follows

$$b = \frac{1}{\eta}\zeta P = -\frac{\sqrt{2}}{2}P.$$

Example 2. *In* $5-$*dimensional Minkowski space* $\mathbf{R}_1^5$*, let*

$$\alpha(t) = \frac{1}{\varepsilon}\left(\kappa, \kappa^2 \sinh \varepsilon t + \varepsilon \tau^2 t, \tau, \kappa\tau \sinh \varepsilon t - \varepsilon\kappa\tau t, \kappa\varepsilon \cosh \varepsilon t\right)$$

be space-like base curve of $3-$*dimensional ruled surface given by*

$$M(t, u_1, u_2) = \alpha(t) + \sum_{v=1}^{2} u_v e_v(t)$$

and time-like subspace $E_2(t) = Sp\{e_1(t), e_2(t)\}$ *be generating space for which*

$$\begin{aligned} e_1(t) &= \tfrac{1}{\varepsilon}\left((\kappa+\tau), \sqrt{3}\kappa \sinh \varepsilon t, (\kappa-\tau), \sqrt{3}\tau \sinh \varepsilon t, \sqrt{3}\varepsilon \cosh \varepsilon t\right) \\ e_2(t) &= \tfrac{1}{\sqrt{3}\varepsilon}\left((\tau-\kappa), \kappa \cosh \varepsilon t, (\kappa+\tau), \tau \cosh \varepsilon t, \varepsilon \sinh \varepsilon t\right) \end{aligned}$$

where κ*,* τ *and* $\varepsilon = \sqrt{\kappa^2+\tau^2}$ *are arbitrary constants. Since* $\langle e_1, e_1\rangle = -1$*, base vector* e_1 *is time-like and since* $\langle e_2, e_2\rangle = 1$*, base vector* e_2 *is space-like. Velocity vector of base curve* α *is*

$$\dot{\alpha}(t) = \left(0, \kappa^2 \cosh \varepsilon t + \tau^2, 0, \kappa\tau \cosh \varepsilon t - \kappa\tau, \kappa\varepsilon \sinh \varepsilon t\right).$$

Therefore we can complete orthonormal base $\{e_1(t), e_2(t)\}$ *of generating space* $E_2(t)$ *of* M *to orthonormal base* $\{e_1(t), e_2(t), a_3(t), a_4(t), a_5(t)\}$ *of* $\mathbf{R}_1^5$ *as*

$$\begin{aligned} a_3(t) &= \tfrac{1}{\sqrt{6}\varepsilon}\left(-(\tau-\kappa), 2\kappa \cosh \varepsilon t, -(\kappa+\tau), 2\tau \cosh \varepsilon t, 2\varepsilon \sinh \varepsilon t\right), \\ a_4(t) &= -\tfrac{1}{\sqrt{2}\varepsilon}\left(\sqrt{3}(\kappa+\tau), 2\kappa \sinh \varepsilon t, \sqrt{3}(\kappa-\tau), 2\tau \sinh \varepsilon t, 2\varepsilon \cosh \varepsilon t\right), \\ a_5(t) &= \tfrac{1}{\varepsilon}(0, \tau, 0, -\kappa, 0). \end{aligned}$$

From these equations we get the following derivative equations

$$\begin{aligned} \dot{e}_1(t) &= \varepsilon e_2(t) + \sqrt{2}\varepsilon a_3(t), \\ \dot{e}_2(t) &= \varepsilon e_1(t) + \sqrt{\tfrac{2}{3}}\varepsilon a_4(t), \\ \dot{a}_3(t) &= \sqrt{2}\varepsilon e_1(t) + \tfrac{2}{\sqrt{3}}\varepsilon a_4(t), \\ \dot{a}_4(t) &= -\sqrt{\tfrac{2}{3}}\varepsilon e_1(t) - \tfrac{2}{\sqrt{3}}\varepsilon a_3(t), \\ \dot{a}_5(t) &= 0. \end{aligned}$$

Thus 1st and 2nd principal Blaschke invariants of $3-$*dimensional time-like ruled surface* M *are calculated to be*

$$b_1 = 0 \quad , \quad b_2 = \frac{\kappa}{\sqrt{2}}$$

and 1st and 2nd principal distribution parameter are found to be

$$P_1 = \frac{\tau}{\sqrt{2}} \quad , \quad P_2 = \sqrt{\frac{3}{2}}\tau.$$

Hence, there exists a relation between principal Blaschke invariants and distribution parameters of M *as follows*

$$b_1 = 0\ P_1 \quad , \quad b_2 = \frac{\kappa}{\sqrt{3}\tau}P_2.$$

$1-$*dimensional subspace* $E(t) = Sp\{e(t)\} \subset E_2(t)$ *is either time-like or space-like subspace. Thus,* $2-$*dimensional ruled surface* $\psi \subset M$ *will be time-like or space-like ruled surface according to whether* $E(t) = Sp\{e(t)\}$ *is time-like or space-like subspace. Now let us calculate these two cases separately.*

1. *If* $E(t) = Sp\{e(t)\} \subset E_2(t)$ *is time-like subspace, then*

$$e(t) = \cosh\theta e_1 + \sinh\theta e_2 \quad , \quad \|e(t)\| = 1.$$

Therefore, the Blaschke invariant of $2-$*dimensional time-like ruled surface* $\psi \subset M$ *is found to be*

$$b = \frac{\kappa \sinh\theta}{\sqrt{\left|9\cosh^2\theta - \sinh^2\theta\right|}}.$$

2. *If* $E(t) = Sp\{e(t)\} \subset E_2(t)$ *is space-like subspace, then*

$$e(t) = \cosh\theta e_1 + \sinh\theta e_2 \quad , \quad \|e(t)\| = 1.$$

Therefore, the Blaschke invariant of $2-$*dimensional space-like ruled surface* $\psi \subset M$ *is found to be*

$$b = \frac{\kappa \sinh\theta}{\sqrt{\left|9\cosh^2\theta - \sinh^2\theta\right|}}.$$

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