Classification of cylindrically symmetric static space-times according to their proper homothetic vector fields

Ghulam Shabbir and M. Ramzan

Abstract. A complete classification of cylindrically symmetric static space-times according to their proper homothetic vector field is given by using direct integration technique. Using the above mentioned technique we have shown that a very special class of the above space-times admit proper homothetic vector field. The dimensions of homothetic Lie algebras are 4, 5, 7 and 11.

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§1. Introduction

The aim of this paper is to study all the possiblilties when the cylindrically symmetric static space-times admit proper homothetic vector fields by using direct integration techinque. Throughout M is representing the four dimensional, connected, hausdorff space-time manifold with Lorentz metric g of signature $(-,+,+,+)$. The curvature tensor associated with g_{ab}, through Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$, and the Ricci tensor components are $R_{ab} = R^c{}_{acb}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively.

Any vector field X on M can be decomposed as

$$X_{a;b} = \frac{1}{2}h_{ab} + F_{ab} \tag{1.1}$$

where $h_{ab} = h_{ba}$ and $F_{ab} = -F_{ba}$ are symmetric and skew-symmetric tensor on M, respectively. If

$$h_{ab} = \alpha g_{ab}, \quad \alpha \in R$$

equivalent,

$$g_{ab,c}X^c + g_{bc}X^c_{,a} + g_{ac}X^c_{,b} = \alpha g_{ab} \tag{1.2}$$

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then X is called a homothetic vector field on M. If X is homothetic and $\alpha \neq 0$ then it is called proper homothetic while $\alpha = 0$ it is Killing [2, 3]. Further consequences and geometrical interpretations of (1.2) are explored in [1, 4]. It also follows from (1.2) that [1, 3]

$$L_X R^a{}_{bcd} = 0 \qquad L_X R_{ab} = 0.$$

The Lie algebra of a set of vector fields on a manifold is completely characterized by the structure constants C^a_{bc} given in term of the Lie brackets by

$$[X_b, X_c] = C^a_{bc} X_a, \quad C^a_{bc} = -C^a_{cb}, \tag{1.3}$$

where X_a are the generators. The Lie algebras for homothetic vector fields in term of structure constants are also given.

§2. Main results

Consider a cylindrically symmetric static space time in the usual coordinate system (labeled as (x^0, x^1, x^2, x^3)) with line element [7]

$$ds^2 = -e^{v(r)}dt^2 + dr^2 + e^{u(r)}d\theta^2 + e^{w(r)}d\phi^2, \tag{2.4}$$

where v, u and w are some functions of r only. The linearly independent Killing vector fields are [5, 6] $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial \phi}$. A vector field X is said to be a homothetic vector field if it satisfies the equation (1.2). One can write (1.2) explicitly using (2.4) as

$$v' X^1 + 2X^0_{,0} = 2c, \tag{2.5}$$

$$X^1_{,0} - e^v X^0_{,1} = 0, \tag{2.6}$$

$$e^u X^2_{,0} - e^v X^0_{,2} = 0, \tag{2.7}$$

$$e^w X^3_{,0} - e^v X^0_{,3} = 0, \tag{2.8}$$

$$X^1_{,1} = c, \tag{2.9}$$

$$e^u X^2_{,1} + X^1_{,2} = 0, \tag{2.10}$$

$$e^w X^3_{,1} + X^1_{,3} = 0, \tag{2.11}$$

$$u' X^0 + 2X^2_{,2} = 2c, \tag{2.12}$$

$$e^w X^3_{,2} + e^u X^2_{,3} = 0, \tag{2.13}$$

$$w^{'} X^0 + 2X^3_{,3} = 2c. \tag{2.14}$$

Equations (2.6), (2.9), (2.10) and (2.11) give

$$\left.\begin{array}{l} X^0 = A^1_t(t,\theta,\phi)\int e^{-v}dr + A^2(t,\theta,\phi) \\ X^1 = cr + A^1(t,\theta,\phi) \\ X^2 = A^1_\theta(t,\theta,\phi)\int e^{-u}dr + A^3(t,\theta,\phi) \\ X^3 = A^1_\phi(t,\theta,\phi)\int e^{-w}dr + A^4(t,\theta,\phi) \end{array}\right\}, \tag{2.15}$$

where $A^1(t,\theta,\phi)$, $A^2(t,\theta,\phi)$, $A^3(t,\theta,\phi)$ and $A^4(t,\theta,\phi)$ are functions of integration. In order to determine $A^1(t,\theta,\phi)$, $A^2(t,\theta,\phi)$, $A^3(t,\theta,\phi)$ and $A^4(t,\theta,\phi)$ we need to integrate the remaining six equations. To avoid lengthy calculations here we will only present results for full details see [8].

Case (1) Four independent homothetic vector fields:

In this case the space-time (2.4) takes the form

$$ds^2 = -(cr+d_9)^{2(1-\frac{d_5}{c})}dt^2 + dr^2 + (cr+d_9)^{2(1-\frac{d_{10}}{c})}d\theta^2 + (cr+d_9)^{2(1-\frac{d_{12}}{c})}d\phi^2, \tag{2.16}$$

and homothetic vector fields in this case are

$$\left.\begin{array}{l} X^0 = td_5 + d_6 \\ X^1 = rc + d_9 \\ X^2 = \theta d_{10} + d_{11} \\ X^3 = \phi d_{12} + d_{13} \end{array}\right\}, \tag{2.17}$$

where $d_5, d_6, d_9, d_{10}, d_{11}, d_{12}, d_{13} \in R$. The above space-time (2.16) admits four independent homothetic vector fields in which three are Killing vector fields which are given in (1.3) and one is proper homothetic vector field which is

$$Y^1 = (0, r, 0, 0). \tag{2.18}$$

The generators in this case are:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \phi} \ and \ X_4 = r\frac{\partial}{\partial r}.$$

Here, all the structure constants are zero.

Case (2) Four independent homothetic vector fields:

In this case the space-time (2.4) takes the form

$$ds^2 = -(cr+d_9)^{2(1-\frac{d_5}{c})}dt^2 + dr^2 + d\theta^2 + (cr+d_9)^{2(1-\frac{d_{12}}{c})}d\phi^2. \tag{2.19}$$

Homothetic vector fields in this case are

$$\left.\begin{array}{l} X^0 = td_5 + d_6 \\ X^1 = rc + d_9 \\ X^2 = \theta c + d_{11} \\ X^3 = \phi d_{12} + d_{13} \end{array}\right\}, \tag{2.20}$$

where $d_5, d_6, d_9, d_{11}, d_{12}, d_{13} \in R$. The above space-time (2.19) admits four independent homothetic vector fields in which three are Killing vector fields and one is proper homothetic vector field which is

$$Y^2 = (0, r, \theta, 0). \tag{2.21}$$

The generators in this case are:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \phi} \textit{ and } X_4 = r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta}.$$

The only non zero structure constant is: $C^2_{24} = 1$

Case (3) Five independent homothetic vector fields:

In this case the space-time (2.4) takes the form

$$ds^2 = (cr + d_9)^{2(1-\frac{d_8}{c})}(-dt^2 + d\phi^2) + dr^2 + (cr + d_9)^{2(1-\frac{d_{10}}{c})}d\theta^2 \tag{2.22}$$

and homothetic vector fields in this case are

$$\left.\begin{array}{l} X^0 = td_8 + \phi d_{13} + d_{15} \\ X^1 = rc + d_9 \\ X^2 = \theta d_{10} + d_{11} \\ X^3 = \phi d_8 + td_{13} + d_{16} \end{array}\right\}, \tag{2.23}$$

where $d_8, d_9, d_{10}, d_{11}, d_{13}, d_{15}, d_{16} \in R$. The above space-time (2.22) admits five independent homothetic vector fields in which four are Killing vector fields and one is proper homothetic vector field which is

$$Y^3 = (0, r, 0, 0). \tag{2.24}$$

The generators in this case are:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \phi}, X_4 = t\frac{\partial}{\partial \phi} + \phi\frac{\partial}{\partial t} \textit{ and } X_5 = r\frac{\partial}{\partial r}.$$

The only non zero structure constant are: $C^3_{14} = C^1_{34} = 1$.

Case (4) Five independent homothetic vector fields:

In this case the space-time (2.4) takes the form

$$ds^2 = (cr + d_9)^{2(1-\frac{d_8}{c})}(-dt^2 + d\phi^2) + dr^2 + d\theta^2 \tag{2.25}$$

and homothetic vector fields in this case are

$$\left.\begin{array}{l} X^0 = td_8 + \phi d_{13} + d_{14} \\ X^1 = rc + d_9 \\ X^2 = \theta c + d_{17} \\ X^3 = \phi d_8 + td_{13} + d_{10} \end{array}\right\}, \tag{2.26}$$

where $d_8, d_9, d_{10}, d_{13}, d_{14}, d_{17} \in R$. The above space-time (2.25) admits five independent homothetic vector fields in which four are Killing vector fields and one is proper homothetic vector field which is

$$Y^4 = (0, r, \theta, 0). \tag{2.27}$$

The generators in this case are:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \phi}, X_4 = t\frac{\partial}{\partial \phi} + \phi\frac{\partial}{\partial t} \text{ and } X_5 = r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta}.$$

The only non zero structure constant are: $C^3_{14} = C^2_{25} = C^1_{34} = 1$.
Case (5) Seven independent homothetic vector fields:

In this case the space-time (2.4) takes the form

$$ds^2 = (cr + d_7)^{2(1-\frac{d_6}{c})}(-dt^2 + d\phi^2 + d\theta^2) + dr^2. \tag{2.28}$$

Homothetic vector fields in this case are

$$\left.\begin{aligned} X^0 &= td_6 + \phi d_{11} + \theta d_{10} + d_{12} \\ X^1 &= rc + d_7 \\ X^2 &= \theta d_6 + td_{10} - \phi d_{13} + d_{15} \\ X^3 &= \phi d_6 + td_{11} + \theta d_{13} + d_{14} \end{aligned}\right\}, \tag{2.29}$$

where $d_6, d_7, d_{10}, d_{11}, d_{12}, d_{13}, d_{14}, d_{15} \in R$. The above space-time (2.28) admits seven independent homothetic vector fields in which six are Killing vector fields and one is proper homothetic vector field which is

$$Y^5 = (0, r, 0, 0). \tag{2.30}$$

The generators in this case are:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \phi}, X_4 = t\frac{\partial}{\partial \phi} + \phi\frac{\partial}{\partial t}, X_5 = t\frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial t}, X_6 = \theta\frac{\partial}{\partial \phi} - \phi\frac{\partial}{\partial \theta}$$

and

$$X_7 = r\frac{\partial}{\partial r}.$$

The only non zero structure constant are: $C^3_{14} = C^2_{15} = C^1_{25} = C^3_{26} = C^1_{34} = C^4_{56} = 1$ and $C^2_{36} = C^6_{45} = C^5_{46} = -1$.
Case (6) Eleven independent homothetic vector fields:

In this case the above space-time (2.4) becomes Minkowski space-time

$$ds^2 = -dt^2 + dr^2 + d\theta^2 + d\phi^2 \tag{2.31}$$

homothetic vector fields in this case are

$$\left.\begin{aligned} X^0 &= tc + rd_{10} + \phi d_{15} + \theta d_{16} + d_{17} \\ X^1 &= rc + td_{10} + \phi d_{11} + \theta d_{12} + d_{13} \\ X^2 &= \theta c - rd_{12} + td_{16} - \phi d_{18} + d_{14} \\ X^3 &= \phi c - rd_{11} + td_{15} + \theta d_{18} + d_{19} \end{aligned}\right\}, \tag{2.32}$$

where $d_{10}, d_{11}, d_{12}, d_{13}, d_{14}, d_{15}, d_{16}, d_{17}, d_{18}, d_{19} \in R$. The above space-time (2.31) admits eleven independent homothetic vector fields in which ten are Killing vector fields and one is proper homothetic vector field which is

$$Y^6 = (t, r, \theta, \phi). \tag{2.33}$$

The generators in this case are:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial r}, X_3 = \frac{\partial}{\partial \theta}, X_4 = \frac{\partial}{\partial \phi}, X_5 = t\frac{\partial}{\partial r} + r\frac{\partial}{\partial t},$$

$$X_6 = t\frac{\partial}{\partial \phi} + \phi\frac{\partial}{\partial t}, X_7 = \theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}, X_8 = \phi\frac{\partial}{\partial r} - r\frac{\partial}{\partial \phi}, X_9 = \theta\frac{\partial}{\partial r} - r\frac{\partial}{\partial \theta},$$

$$X_{10} = \theta\frac{\partial}{\partial \phi} - \phi\frac{\partial}{\partial \theta} \textit{ and } X_{11} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta} + \phi\frac{\partial}{\partial \phi}.$$

The only non zero structure constant are: $C^1_{15} = C^4_{16} = C^3_{17} = C^1_{11} = C^1_{25} = C^2_{21} = C^1_{37} = C^2_{39} = C^4_{31} = C^3_{31} = C^1_{46} = C^2_{48} = C^4_{41} = C^5_{68} = C^5_{79} = C^6_{71} = C^1_{89} = C^8_{91} = 1$ and $C^4_{28} = C^3_{29} = C^3_{41} = C^8_{56} = C^9_{57} = C^6_{58} = C^7_{59} = C^1_{67} = C^7_{61} = C^9_{81} = -1$.

SUMMARY

In this paper a study of cylinderically symmetric static space-times according to their proper homothetic vector fields is given by using the direct integration technique. From the above study we obtain the following:

(i) The space-times which admit four independent homothetic vector fields are given in equations (2.16) and (2.19) (see for details cases (1) and (2)).

(ii) The space-times which admit five independent homothetic vector fields are given in equations (2.22) and (2.25) (see for details cases (3) and (4)).

(iii) The space-time which admits seven independent homothetic vector fields are given in equation (2.28) (see for details case (5)).

(iv) The space-time which admits eleven independent homothetic vector fields are given in equation (2.31) (see for details case (6)).

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Authors' addresses:

Ghulam Shabbir and M. Ramzan
Faculty of Engineering Sciences,
GIK Institute of Engineering Sciences and Technology,
Topi, Swabi, NWFP, Pakistan.
e-mail: shabbir@giki.edu.pk