More on nonconvexities in an optimal growth model

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Abstract. In an endogenous growth model with marginal product of capital minus population growth rate monotonically decreasing in capital, we show that the economy has a unique balanced growth path (BGP) equilibrium which is saddle-point stable. Moreover, along a BGP path, when all exogenous factors are controlled for, there exists a negative relation between fertility growth and economic growth as well as between fertility and environmental quality.

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§1. Introduction

Economists have been interested in dynamic problems since at least the works of Evans [6], Hotelling [8] and Ramsey [11] in the 1920s. However, it was only after the 1960s that dynamic mathematical techniques were widely introduced into economics. The methodology used by classical mathematicians, known as the calculus of variations, has since been generalized in two ways. First, by Bellman [4] who developed the method of dynamic programming. Second, by Pontryagin et al. [10] who developed the Maximum principle of optimal control, which furnishes a set of necessary conditions for optimality. In general, these conditions are not sufficient for maximization, unless certain concavity conditions are satisfied. It was showed by Arrow [2] that if the maximized current-value Hamiltonian is concave in state variables, given co-state variables, then the first order conditions, together with a transversality condition, are necessary and sufficient to characterize the maximum.

In this paper we consider a model with an endogenous fertility choice, where the number of children enters directly into the utility function of consumers, along with consumption and environmental quality. In this regard, we employ the standard practice of the literature on optimizing real growth models with endogenous fertility, see e.g. [3], [5], [9], [14], and [15]. Since there is an endogenous fertility choice, the production possibilities set is not convex. In other words, with the population growth rate n, as a control variable, nonlinearities and nonconvexities are introduced into the budget constraint, and so the sufficiency of the necessary conditions are not guaranteed. To assure it, some restrictions have to imposed. The assumption that the marginal product of capital minus the population growth rate is a monotone decreasing function of the capital stock provides a condition for that. Within this framework,

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we prove that there exists a unique balanced growth path (BGP) equilibrium, which is saddle-point stable. Furthermore, an analysis of the relationship between fertility and economic growth, as well as between fertility and environmental quality, along a BGP equilibrium shows the existence of an inverse relationship between them when all exogenous factors are controlled for. This work generalizes Germanà and Guerrini's paper [7] through the assumption of a more general utility function, technology, and time cost of child-rearing.

§2. The model

We consider a closed economy inhabited by identical agents who are infinitely lived. Agent's preferences are represented by an instantaneous utility function u which depends on per capita consumption c, on population growth rate n, and on environmental quality E, that is $u(c(t), n(t), E(t))$, where t is a time index. The insertion of the fertility rate into the utility function of the representative agent allows fertility to be endogenously chosen. Given a zero growth death rate and the absence of immigration, the fertility rate coincides with population growth.

Each agent has access to a technology described by a production function f such that

$$y = f(k, \bar{k}l), \tag{2.2.1}$$

where y, k, l denote respectively output, capital and time spent producing goods, all in per capita terms. In (2.1) time index has been dropped to simplify the notation. The average stock of capital, $\bar{k} > 0$, yields an externality such that in equilibrium, when $k = \bar{k}$, the production function is linear in accumulating stock of capital as in Romer [12]. The function f is assumed to be twice continuously differentiable, linearly homogeneous, strictly increasing and strictly concave with respect to k and l, and such that the Inada conditions hold. Observe that a consequence of the constant returns to scale assumption on f is that $f_{kl} > 0$.

Each agent is endowed with a unit of time that can be allocated either to child-rearing $\phi(n)$ or to labor l, that is

$$\phi(n) + l = 1. \tag{2.2.2}$$

The function $\phi(n)$ is postulated to be twice continuously differentiable, strictly increasing in n, with $\phi(0) = 0$ and $\phi(1/\delta) = 1$, where $1/\delta > 0$ represents the fertility limit which agents can reach. The second derivative of $\phi(n)$ can be of either sign, implying that the marginal time cost of taking care of children can be either increasing, constant, or decreasing. For any given value of k we impose that the opportunity cost of children in terms of output, $\phi'(n)f_l(k, \bar{k}(1 - \phi(n))) + k$, is strictly increasing in n, that is

$$\phi''(n)f_l(k, \bar{k}(1 - \phi(n))) - \bar{k}(\phi'(n))^2 f_{ll}(k, \bar{k}(1 - \phi(n))) > 0. \tag{2.2.3}$$

This condition is automatically fulfilled if there is a constant or increasing marginal cost of child rearing, that is if $\phi''(n) \geq 0$.

Environmental quality, which is measured by the rate of deforestation, is assumed to have a finite upper bound. Hence, E is negative being measured by the difference between the actual level and this upper limit. As in Aghion and Howitt [1], we suppose that there exists a finite lower limit for E, say E_{min}, under which there will be a catastrophe. Thus, $E_{min} \leq E \leq 0$. The evolution of environmental quality is described by

$$\dot{E} = -\eta E - \theta n, \tag{2.2.4}$$

where $\eta \in (0,1)$ characterizes the capacity of natural regeneration of the environment, $\theta > 0$ measures the importance of the environmental destruction due to demographic pressures, and a dot over the variable E denotes time derivative.

Regarding the utility function u, we assume that it is additively separable in consumption, on the one hand, and fertility and environmental quality, on the other, that is

$$u(c, n, E) = \ln(c) + v(n, E).$$

The function v is postulated to be nonnegative, twice continuously differentiable, strictly increasing and strictly concave in its arguments, with the partial derivative v_n increasing in E, and such that

$$\lim_{n \to 0} v_n(n, E) = \infty, \quad \lim_{n \to 1/\delta} v_n(n, E) < \infty,$$

and

$$\lim_{E \to 0} v_E(n, E) = 0, \quad \lim_{E \to E_{\min}} v_E(n, E) < \infty.$$

Given the initial capital stock, $k(0) > 0$, the representative agent seeks to maximize the following intertemporal utility function

$$\int_0^\infty u(c, n, E)\, e^{-\rho t} dt$$

subject to the constraints (2.2), (2.4) and the flow budget constraint

$$c + \dot{k} + nk = f(k, \bar{k}l). \tag{2.2.5}$$

$\rho > 0$ denotes the constant rate of time preference. To derive the necessary conditions for an optimal policy, we apply the Pontryagin maximum principle. The current-value Hamiltonian is defined as

$$H(c, n, k, E, \lambda, \mu) = u(c, n, E) + \lambda[f(k, \bar{k}(1 - \phi(n))) - nk - c] + \mu[-\eta E - \theta n],$$

where μ and λ denote the co-state variables associated with the constraints (2.4) and (2.5), respectively. Maximizing H with respect to c and n yields

$$H_c = 0 \quad \Rightarrow \quad c^{-1} = \lambda \tag{2.2.6}$$

$$H_n = 0 \quad \Rightarrow \quad v_n(n, E) = \lambda[\bar{k}\phi'(n) f_l(k, \bar{k}(1 - \phi(n))) + k] + \mu\theta \tag{2.2.7}$$

where a subindex denotes the variable with respect to which the partial derivative is taken.

¿From the Pontryagin maximum principle, we also obtain the following differential equations, which govern the behaviour of λ and μ,

$$\dot{\lambda} = \rho\lambda - H_k \quad \Rightarrow \quad \dot{\lambda} = \rho\lambda - \lambda[f_k(k, \bar{k}(1 - \phi(n))) - n] \tag{2.2.8}$$

$$\dot{\mu} = \rho\mu - H_E \quad \Rightarrow \quad \dot{\mu} = (\rho + \eta)\mu - v_E(n, E) \tag{2.2.9}$$

plus the transversality condition at infinity

$$\lim_{t\to\infty} (e^{-\rho t}\lambda k + e^{-\rho t}\mu E) = 0. \tag{2.2.10}$$

Equations (2.6) – (2.10) constitute the set of necessary conditions for the optimum. In order to proceed our analysis, we will express the variables c and n in terms of k, E, λ, and μ. From (2.6) it is immediate that

$$c = c(\lambda). \tag{2.2.11}$$

Next, we set

$$F(n, k, E, \lambda, \mu) = v_n(n, E) - \lambda\bar{k}\phi'(n)f_l(k, \bar{k}(1 - \phi(n))) - \lambda k - \mu\theta. \tag{2.2.12}$$

¿From (2.7) we find that $F(n, k, E, \lambda, \mu) = 0$. Since

$$\begin{aligned} F_n(n, X, E, \mu) &= v_{nn}(n, E) \\ &\quad - \lambda\bar{k}[\phi''(n)f_l(k, \bar{k}(1 - \phi(n))) - \bar{k}(\phi'(n))^2 f_{ll}(k, \bar{k}(1 - \phi(n)))] < 0, \end{aligned} \tag{2.2.13}$$

an application of the Implicit function theorem yields

$$n = n(k, E, \lambda, \mu) \text{ and } F(n(k, E, \lambda, \mu), k, E, \lambda, \mu) = 0. \tag{2.2.14}$$

Assumption (2.3) together with $u_{cc} < 0$ ensure that the Hessian of the current-value Hamiltonian is negative definite with respect to c and n, which is a sufficient condition for a maximum of H. We define the maximized current-value Hamiltonian as

$$\begin{aligned} H^0(k, E, \lambda, \mu) &= \max_{c,n} H(c, n, k, E, \lambda, \mu) \\ &= H(c(\lambda), n(k, E, \lambda, \mu), k, E, \lambda, \mu). \end{aligned} \tag{2.2.15}$$

Replacing c and n in (2.15) with their expressions derived in (2.11) and (2.14), we have

$$\begin{aligned} H^0(k, E, \lambda, \mu) = \ln c(\lambda) + v(n(k, E, \lambda, \mu), E) + \lambda[f(k, \bar{k}(1 - \phi(n(k, \lambda, \mu)))) \\ - n(k, E, \lambda, \mu)k - c(\lambda)] + \mu[-\eta E - \theta n(k, \lambda, \mu)]. \end{aligned}$$

Differentiating the latter with respect to k, and using (2.6) and (2.7), we obtain

$$H^0_k(k, E, \lambda, \mu) = \lambda[f_k(k, \bar{k}(1 - \phi(n(k, E, \lambda, \mu)))) - n(k, E, \lambda, \mu)]. \tag{2.2.16}$$

Proposition 1. *The first order necessary conditions for optimality are also sufficient if the marginal product of capital minus the population growth rate is monotonically decreasing in k, that is if*

$$\frac{\partial}{\partial k}[f_k(k, \bar{k}(1-\phi(n))) - n] < 0. \tag{2.2.17}$$

Proof. Assumption (2.17) together with (2.16) imply that $H^0(k, E, \lambda, \mu)$ is concave with regard to k, given E, λ and μ. Moreover, as

$$H^0_{EE}(k, E, \lambda, \mu) = v_{EE}(n, E) < 0,$$

it is also concave with respect to E. An application of Arrow's theorem [2] gives the result. □

§3. Equilibrium and stability analysis

In this section we introduce the variable $X = c^{-1}k$ and use the equilibrium condition $k = \bar{k}$. Recalling that if a function is homogeneous of degree 1, then its first partial derivatives are homogeneous of degree 0, we have that (2.12) and (2.13) rewrite as

$$F(n, X, E, \mu) = v_n(n, E) - X[\phi'(n) f_l(1, 1-\phi(n)) + 1] - \mu\theta,$$

and

$$F_n = v_{nn}(n, E) - X[\phi''(n) f_l(1, 1-\phi(n)) - (\phi'(n))^2 f_{ll}(1, 1-\phi(n))] < 0.$$

Since

$$F(n, X, E, \mu) = 0, \tag{3.3.18}$$

the Implicit function theorem ensures the existence of a function

$$n = n(X, E, \mu)$$

such that

$$F(n(X, E, \mu), X, E, \mu) = 0.$$

Furthermore, a simple differentiation of this equation yields

$$\begin{aligned} n_X(X, E, \mu) &= -\frac{F_X(n(X, E, \mu), X, E, \mu)}{F_n(n(X, E, \mu), X, E, \mu)} \\ n_E(X, E, \mu) &= -\frac{F_E(n(X, E, \mu), X, E, \mu)}{F_n(n(X, E, \mu), X, E, \mu)} \\ n_\mu(X, E, \mu) &= -\frac{F_\mu((n(X, E, \mu), X, E, \mu)}{F_n(n(X, E, \mu), X, E, \mu)} \end{aligned}$$

where $F_X < 0$, $F_\mu < 0$, and $F_E \geq 0$. Consequently, we have proved the following result.

Lemma 1. *Let* $X = c^{-1}k$. *Then*

$$n = n(X, E, \mu), \text{ with } n_X(X, E, \mu) < 0,\ n_E(X, E, \mu) \geq 0 \text{ and } n_\mu(X, E, \mu) < 0.$$

The introduction of the new variable X allows to reduce the dynamical system (2.6) – (2.9) to a system of three differential equations in (X, E, μ).

Proposition 2. *The dynamical system*

$$\begin{aligned} \dot{X} &= f(1, 1-\phi(n(X,E,\mu)))X + f_k(1, 1-\phi(n(X,E,\mu)))X + \rho X - 1 \\ \dot{E} &= -\eta E - \theta n(X,E,\mu) \\ \dot{\mu} &= (\rho+\eta)\mu - v_E(n(X,E,\mu), E) \end{aligned}$$

together with the transversality condition $\lim_{t\to\infty} (e^{-\rho t}X + e^{-\rho t}\mu E) = 0$, *describes the dynamic behaviour in the* (X, E, μ) *space of the system of necessary conditions for an optimal program.*

Proof. It remains only to prove the law motion of X. This follows immediately by differentiating $X = c^{-1}k$ with respect to time and using equations (2.6) and (2.8) with the condition $k = \bar{k}$. □

We recall a balanced growth path (BGP henceforth) equilibrium to be a set of paths $\{c, n, k, E, \mu\}$, which solves the optimal control problem, such that the variables c and k grow at the same constant rate, E and μ are constant, and $n = n(X, E, \mu)$.

Proposition 3. *If condition* (2.17) *holds, the model has a unique BGP equilibrium which is saddle-point stable.*

Proof. First, since $\dot{k}/k = \dot{c}/c$ on the BGP, X is also constant. So $\lim_{t\to\infty} e^{-\rho t}X = 0$. Furthermore, since E and μ are constant, $\lim_{t\to\infty} e^{-\rho t}\mu E = 0$. Hence, the transversality condition (2.10) is satisfied. Now balanced growth equilibrium is attained when

$$\dot{X} = \dot{E} = \dot{\mu} = 0$$

and is therefore characterized by the following conditions

$$\begin{aligned} X &= \frac{1}{f(1, 1-\phi(n(X,E,\mu))) + f_k(1, 1-\phi(n(X,E,\mu))) + \rho} \\ E &= -\frac{\theta n(X,E,\mu)}{\eta} \\ \mu &= \frac{v_E(n(X,E,\mu), -\theta n(X,E,\mu)/\eta)}{\rho+\eta}. \end{aligned}$$

¿From these and (3.1) we find that

$$v_n(n, -\theta n/\eta) = \frac{\phi'(n) f_l(1, 1-\phi(n)) + 1}{f(1, 1-\phi(n)) + f_k(1, 1-\phi(n)) + \rho} + \frac{\theta v_E(n, -\theta n/\eta)}{\rho+\eta}.$$

The left hand side of this equality is a function strictly decreasing in n. Its right hand side is instead a function strictly increasing in n. Hence, taking into account the behaviour at the boundary of the functions v and ϕ, we see that this equation has a unique solution n^*, and, consequently, there exist unique X^*, E^* and μ^*. This proves the first statement of the Proposition. The purpose is now to analyze the eigenvalue structure of the Jacobian matrix corresponding to the linearization of the above dynamical system around the steady state equilibrium (X^*, E^*, μ^*). Hence, we first characterize this Jacobian matrix evaluated at the equilibrium point, say J^*. This takes the form

$$J^* = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix},$$

where

$$\begin{aligned}
J_{11}^* &= \frac{\partial \dot{X}}{\partial X}\,|_{(X^*,E^*,\mu^*)} &&= f^* + f_k^* + \rho + [f_l^* + f_{kl}^*]\phi'(n^*)X^*(-n_X^*) \\
J_{12}^* &= \frac{\partial \dot{X}}{\partial E}\,|_{(X^*,E^*,\mu^*)} &&= [f_l^* + f_{kl}^*]\phi'(n^*)X^*(-n_E^*) \\
J_{13}^* &= \frac{\partial \dot{X}}{\partial \mu}\,|_{(X^*,E^*,\mu^*)} &&= [f_l^* + f_{kl}^*]\phi'(n^*)X^*(-n_\mu^*) \\
J_{21}^* &= \frac{\partial \dot{E}}{\partial X}\,|_{(X^*,E^*,\mu^*)} &&= \theta(-n_X^*) \\
J_{22}^* &= \frac{\partial \dot{E}}{\partial E}\,|_{(X^*,E^*,\mu^*)} &&= -\eta + \theta(-n_E^*) \\
J_{23}^* &= \frac{\partial \dot{E}}{\partial \mu}\,|_{(X^*,E^*,\mu^*)} &&= \theta(-n_\mu^*) \\
J_{31}^* &= \frac{\partial \dot{\mu}}{\partial X}\,|_{(X^*,E^*,\mu^*)} &&= v_{En}^*(-n_X^*) \\
J_{32}^* &= \frac{\partial \dot{\mu}}{\partial E}\,|_{(X^*,E^*,\mu^*)} &&= -v_{EE}^* + v_{En}^*(-n_E^*) \\
J_{33}^* &= \frac{\partial \dot{\mu}}{\partial \mu}\,|_{(X^*,E^*,\mu^*)} &&= \rho + \eta + v_{En}^*(-n_\mu^*).
\end{aligned}$$

In the above, functions beneath overstars are evaluated at the stationary values, that is $n^* = n(X^*, E^*, \mu^*)$, $f^* = f(1, 1 - \phi(n^*))$, $v_{En}^* = v_{En}(n^*, E^*)$, and so on. The signs of the eigenvalues of the matrix J^* can be derived looking at the trace and the determinant of J^*. The trace of J^* is given by

$$Trace(J^*) = J_{11}^* + J_{22}^* + J_{33}^* = f^* + f_k^* + 2\rho + [f_l^* + f_{kl}^*]\phi'(n^*)X^*(-n_X^*).$$

As $n_X^* < 0$, the sign of $Trace(J^*)$ is positive. Since this trace is also equal to the sum of the eigenvalues of J^*, there must be at least one eigenvalue with positive real part.

On the other hand, the determinant of J^* is given by

$$Det(J^*) = (f^* + f_k^* + \rho)[-\eta(\rho+\eta) + \eta v_{En}^* n_\eta^* - (\rho+\eta)\theta n_E^* - \theta v_{EE}^* n_\eta^*] \\ + [f_l^* + f_{kl}^*]\phi'(n^*)X^*\eta(\rho+\eta)n_X^*.$$

As $v_{En}^* \geq 0$, $v_{EE}^* < 0$, $n_X^* < 0$, $n_E^* \geq 0$, and $n_\mu^* < 0$, the sign of $Det(J^*)$ is negative. Since $Det(J^*)$ is the product of the eigenvalues of J^*, this leaves two possibilities open for the eigenvalues of J^*. They can be either three eigenvalues with negative real parts or one negative and two with positive real parts (recall that complex eigenvalues must occur as conjugate pairs). Since we already know that there cannot be three eigenvalues with negative real part, we can exclude the first case. This implies that the stationary solution (X^*, E^*, μ^*) is locally saddle-point stable. □

§4. Fertility, growth and environmental quality

We are going to examine the relationship between population growth, economic growth and environmental quality. Along a balanced growth path c and k grow at the same constant rate, say g^*. Therefore, it follows from (2.6) and (2.8) that

$$g^* = f_k(1, 1-\phi(n^*)) - n^* - \rho. \tag{4.4.19}$$

Hence, by differentiating (4.1) with respect to n^* we find that

$$\frac{\partial g^*}{\partial n^*} = -[1 + \phi'(n^*) f_{kl}(1, 1-\phi(n^*))] < 0.$$

We have proved the following result.

Proposition 4. *There exists an inverse relationship between economic growth and population growth when all exogenous factors are controlled for.*

Regarding the relationship between population growth and environmental quality, from being $E^* = -\theta n^*/\eta$ at the BGP equilibrium, we can conclude that an increase in population growth damages environmental quality. High values of n^* will lead E^* to reach its lower limit E_{min}, and consequently to an environmental catastrophe.

In our analysis of how the BGP equilibrium values react to exogenous parameter change, we now examine the case when some of the exogenous factors are not controlled for, but are allowed to vary. For example, let consider an improvement in technological progress. For simplicity, we assume the production function f to be Cobb-Douglas, i.e. of the form $f(k, \bar{k}l) = Ak^\alpha l^{1-\alpha}\bar{k}^{1-\alpha}$, where $A > 0$ is a scale parameter and $0 < \alpha < 1$. Since (4.1) writes as

$$g^* = A\alpha(1-\phi(n^*))^{1-\alpha} - n^* - \rho,$$

we get the following comparative static result

$$\frac{\partial g^*}{\partial A} = \alpha(1-\phi(n^*))^{1-\alpha} - [\alpha(1-\alpha)A\phi'(n^*)(1-\phi(n^*))^{-\alpha} + 1]\,\frac{\partial n^*}{\partial A}. \tag{4.4.20}$$

Next, (2.4), (2.5) and (3.1) along the BGP equilibrium imply that

(4.4.21) $$g^* = A(1-\phi(n^*))^{1-\alpha} - n^* - (X^*(n))^{-1}.$$

with

$$X^* = X^*(n^*) = \frac{v_n(n^*, -\theta n^*/\eta) - \dfrac{\theta v_E(n^*, -\theta n^*/\eta)}{\rho+\eta}}{A(1-\alpha)(1-\phi(n^*))^{-\alpha}\phi'(n^*)+1}.$$

Hence, by differentiating (4.3) with respect to A and using (4.2), we get

$$\frac{\partial n^*}{\partial A} = \frac{(1-\alpha)(1-\phi(n^*))^{1-\alpha}}{A(1-\alpha)^2(1-\phi(n^*))^{-\alpha}\phi'(n^*) - X'(n^*)/X(n^*)^2}.$$

Now $\partial n^*/\partial A > 0$ since $X'(n^*) > 0$, while $\partial g^*/\partial A$ is ambiguous in sign. Furthermore, we have that $\partial E^*/\partial A = -(\theta/\eta)\partial n^*/\partial A < 0$. Consequently, we can conclude that the relation between n^* and g^* is generally indeterminate, while, concerning environmental quality, we have that the technological progress implies an increase in population growth rate and a degradation of environmental quality.

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