Folding on the Cartesian product of manifolds and their fundamental group

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Abstract. In this article, we introduce the fundamental group of foldings of the Cartesian product of manifolds into itself. Also the fundamental group of the limit of foldings of the Cartesian product of manifolds into itself are deduced. The effect of folding on the wedge sume of manifolds and their fundamental group will be achieved. Some types of conditional foldings restricted on the elements of a free group and their fundamental groups are represented. Theorems governing these relations are obtained.

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Key words: manifold, folding, fundamental group, Cartesian product.

§1. Introduction

In this article, the concept of foldings will be discussed from viewpoint of algebra. The effect of foldings on the manifold M or on a finite number of product manifolds $M_1 \times M_2 \times ... \times M_n$ on the fundamental group $\pi_1(M)$ and $\pi_1(M_1 \times M_2 \times ... \times M_n)$ will be presented. The algebraic studies on every manifolds has an algebraic structure if the manifold changed into another manifold by some transformations, then the algebraic structure of the first one is different from the algebraic structure of the second. One of the main techniques of algebraic topology is to study topological spaces by forming algebraic images of them. Most often these algebraic images are groups, but more elaborate structures such as rings, modules, and algebras also arise. The mechanisms which creat these images the 'lanterns' of algebraic topology, one might say are known formally as functors and have the characteristic feature that they form images not only of spaces but also maps. Thus, continuous maps between spaces are projected onto homomorphisms between their algebraic images, so topologically related spaces have algebraically related images [11]. The folding of a manifold was firstly introduced by Robertson 1977 [14]. More studies of the folding of many types of manifolds were studied in [2-4,6-9]. The unfolding of a manifold introduced in [5]. Some applications of the folding of a manifold discussed in [1]. The fundamental groups of some types of a manifold are discussed in [10-13].

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§2. Definitions

1. The set of homotopy classes of loops based at the point x_0 with the product operation $[f][g] = [f.g]$ is called the fundamental group and denoted by $\pi_1(X, x_0)$ [11].

2. Let M and N be two manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma : I \to M$, the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ [14]. If f does not preserve length, it is called a topological folding [9].

3. Let M and N be two manifolds of the same dimension. A map $g : M \to N$ is said to be unfolding of M into N if, for every piecewise geodesic path $\gamma : I \to M$, the induced path $g \circ \gamma : I \to N$ is piecewise geodesic with length greater than γ [5].

4. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained by identifying x_0 and y_0 to a single point [11].

§3. The main results

Theorem 1. *If $M_1, M_2, ..., M_n$ are path connected manifolds and F is a folding from $\overset{n}{\underset{i=1}{\vee}} M_i$ into itself then there is an induced folding $\overline{F}$ of $\overset{n}{\underset{i=1}{*}} \pi_1(M_i)$ into itself which reduce the degree of $\overset{n}{\underset{i=1}{*}} \pi_1(M_i)$.*

Proof. Let $F : \overset{n}{\underset{i=1}{\vee}} M_i \to \overset{n}{\underset{i=1}{\vee}} M_i$ be folding on $\overset{n}{\underset{i=1}{\vee}} M_i$ into itsself.

Then $F : \overset{n}{\underset{i=1}{\vee}} M_i \to \overset{n}{\underset{i=1}{\vee}} M_i$ has the following forms.

If $F(\overset{n}{\underset{i=1}{\vee}} M_i) = M_1 \vee M_2 \vee ... \vee F(M_s) \vee ... \vee M_n$ for $s = 1, 2, ..., n$, then

$\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1(M_i)) = \pi_1(F(\overset{n}{\underset{i=1}{\vee}} M_i)) \approx \pi_1(M_1) * \pi_1(M_2) * ... * \pi_1(F(M_s)) * ... * \pi_1(M_n)$.
Since $\deg ree(\pi_1(F(M_s))) \leq \deg ree(\pi_1(M_s))$ it follows that $\overline{F}$ reduce the degree of $\overset{n}{\underset{i=1}{*}} \pi_1(M_i)$.

Also, if $F(\overset{n}{\underset{i=1}{\vee}} M_i) = M_1 \vee M_2 \vee .. \vee F(M_s) \vee ... \vee F(M_k) \vee ... \vee M_n$ for $k = 1, 2, ..., n$, $s < k$ then $\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1(M_i)) = \pi_1(F(\overset{n}{\underset{i=1}{\vee}} M_i)) \approx \pi_1(M_1) * \pi_1(M_2) * ... * \pi_1(F(M_s)) * ... * \pi_1(F(M_k)) * ... * \pi_1(M_n)$ and so $\overline{F}$ reduce the degree of $\overset{n}{\underset{i=1}{*}} \pi_1(M_i)$. Moreover, by continuing this process if $F(\overset{n}{\underset{i=1}{\vee}} M_i) = \overset{n}{\underset{i=1}{\vee}} F(M_i)$.

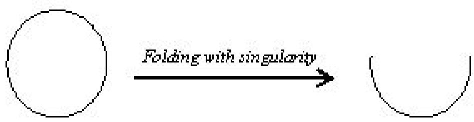
Then $\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1(M_i)) = \pi_1(F(\overset{n}{\underset{i=1}{\vee}} M_i)) = \pi_1(\overset{n}{\underset{i=1}{\vee}} F(M_i)) \approx \overset{n}{\underset{i=1}{*}} \pi_1(F(M_i))$.

Hence $\overline{F}$ reduce the degree of $\overset{n}{\underset{i=1}{*}} \pi_1(M_i)$. □

Theorem 2. *For every $k \leq n$,there is a folding F_k of $\overset{n}{\underset{i=1}{\vee}} S_i^1$ into itself which induces a folding $\overline{F_k}$ of $\overset{n}{\underset{i=1}{*}} \pi_1(S_i^1)$ into itself such that $\overline{F_k}(\overset{n}{\underset{i=1}{*}} \pi_1(S_i^1))$ is a free group of rank $n-k$.*

Proof. Let $F_1 : \overset{n}{\underset{i=1}{\vee}} S_i^1 \to \overset{n}{\underset{i=1}{\vee}} S_i^1$ be a folding such that
$S_1^1 \vee S_2^1 \vee ... \vee F_1(S_t^1) \vee ... \vee S_n^1$ for $t = 1, 2, ..., n$ and $F_1(S_t^1) \neq S_t^1$ folding with singularity as in FIGURE 1, then there is an induced folding $\overline{F_1} : \overset{n}{\underset{i=1}{*}} \pi_1(S_i^1) \to \overset{n}{\underset{i=1}{*}} \pi_1(S_i^1)$ such that $\overline{F_1}(\overset{n}{\underset{i=1}{*}} \pi_1(S_i^1)) = \pi_1(F_1(\overset{n}{\underset{i=1}{\vee}} S_i^1))$ and so $\overline{F_1}(\overset{n}{\underset{i=1}{*}} \pi_1(S_i^1)) \approx \pi_1(S_1^1) * \pi_1(S_2^1) * ... * \pi_1(F_1(S_t^1)) * ... * \pi_1(S_n^1)$. Since $\pi_1(F_1(S_t^1)) = 0$ and $\pi_1(S_i^1) \approx Z$, it follows that $\overline{F_1}(\overset{n}{\underset{i=1}{*}} \pi_1(S_i^1))$ is a free group of rank $n-1$. Also, let $F_2 : \overset{n}{\underset{i=1}{\vee}} S_i^1 \to \overset{n}{\underset{i=1}{\vee}} S_i^1$ be a folding such that $F_2(\overset{n}{\underset{i=1}{\vee}} S_i^1) = S_1^1 \vee S_2^1 \vee ... \vee F_2(S_s^1) \vee ... \vee F_2(S_t^1) ... \vee S_n^1$ for $s, t = 1, 2, ..., n, s < t$ and $F_2(S_s^1) \neq S_s^1, F_2(S_t^1) \neq S_t^1$ then we can get the induced folding $\overline{F_2} : \overset{n}{\underset{i=1}{*}} \pi_1(S_i^1) \to \overset{n}{\underset{i=1}{*}} \pi_1(S_i^1)$ such that $\overline{F_2}(\overset{n}{\underset{i=1}{*}} \pi_1(S_i^1))$ is a free group of rank $n-2$.

By continuing this process we obtain the folding $F_n : \overset{n}{\underset{i=1}{\vee}} S_i^1 \to \overset{n}{\underset{i=1}{\vee}} S_i^1$such that $F_n(\overset{n}{\underset{i=1}{\vee}} S_i^1) = \overset{n}{\underset{i=1}{\vee}} F_n(S_i^1)$ and $F_n(S_i^1) \neq S_i^1$ which induces a folding $\overline{F_n} : \overset{n}{\underset{i=1}{*}} \pi_1(S_i^1) \to \overset{n}{\underset{i=1}{*}} \pi_1(S_i^1)$ such that $\overline{F_n}(\overset{n}{\underset{i=1}{*}} \pi_1(S_i^1))$ is a free group of rank 0. □



Theorem 3. *Let $D_n, n \geq 2$ be the disjoint union of n discs on the sphere S^2 and $\{F_m, m \in N\}$ be a sequence of conditional folding from $S^2 - D_n$ into itself then there is an induced folding $\overline{F}_m : \pi_1(S^2 - D_n) = \pi_1(S^2 - D_n)$ which depends on the conditional folding F_m such that $\pi_1(\lim\limits_{m\to\infty}(F_m(S^2 - D_n))$ is a free group of rank $n-2$.*

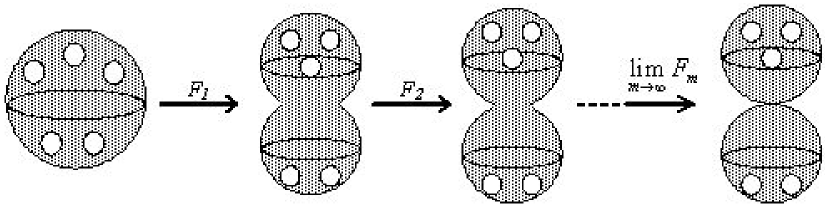
Proof. Let $D_n, n \geq 2$ be the disjoint union of n discs on the sphere S^2 then we can define a squence of foldings $F_m : S^2 - D_n \to S^2 - D_n, m = 1, 2, ...$ such that $\lim\limits_{m\to\infty}(F_m(S^2 - D_n) = (S^2 - D_h) \vee (S^2 - D_k)$ where $k + h = n$ as in FIGURE 2 thus $\pi_1(\lim\limits_{m\to\infty}(F_m(S^2 - D_n)) = \pi_1(S^2 - D_h) * \pi_1(S^2 - D_k)$ and so

$\pi_1(\lim\limits_{m\to\infty}(F_m(S^2 - D_n)) \approx \underbrace{Z * Z * ... * Z}_{h-1} * \underbrace{Z * Z * ... * Z}_{k-1}$.

Hence, $\pi_1(\lim\limits_{m\to\infty}(F_m(S^2 - D_n)) \approx \underbrace{Z * Z * ... * Z}_{h+k-1}$.

Therefore, $\pi_1(\lim\limits_{m\to\infty}(F_m(S^2 - D_n))$ is a free group of rank $n-2$.

□



Theorem 4. *If $M_1, M_2, ..., M_n$ are path connected manifolds and F is a folding from $\prod_{i=1}^{n} M_i$ into itself then there is an induced folding $\overline{F}$ of $\pi_1(\prod_{i=1}^{n} M_i)$ into itself such that $\overline{F}(\pi_1(\prod_{i=1}^{n} M_i)) \approx \pi_1(M_1) \times \pi_1(M_2) \times ... \times \pi_1(F(M_s)) \times ... \times \pi_1(M_n)$ for $s = 1, 2, ..., n$ or $\approx \pi_1(M_1) \times \pi_1(M_2) \times ... \times \pi_1(F(M_s)) \times ... \times \pi_1(F(M_k)) \times ... \times \pi_1(M_n)$ for $s, k = 1, 2, ..., n, s < k, ...,$ or $\approx \pi_1(\prod_{i=1}^{n} F(M_i))$.*

Proof. Let $F : \prod_{i=1}^{n} M_i \to \prod_{i=1}^{n} M_i$ be folding from $\prod_{i=1}^{n} M_i$ into itself, then F is continuous map. So we have the coordinate system of $\prod_{i=1}^{n} M_i$ will be on the form $\{(U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_n}), (X_{\alpha_1} \times X_{\alpha_2} \times ... \times X_{\alpha_n})\}$, where X_{α_i} is injective and bicontinous mapping from an open subset form $U_{\alpha_i} \subseteq R^{n_i} \to M_i$ for $i = 1, 2, ..., n$ and $\{(U_{\alpha_i}, X_{\alpha_i})\}$ is the atlas of M_i for $i = 1, 2, ..., n$ then $F : \prod_{i=1}^{n} M_i \to \prod_{i=1}^{n} M_i$ has the following forms: If $F(\prod_{i=1}^{n} M_i) = F(U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_n}, X_{\alpha_1} \times X_{\alpha_2} \times ... \times X_{\alpha_n}) = (U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_n}, F(U_{\alpha_s}, X_{\alpha_s}), X_{\alpha_1} \times X_{\alpha_2} \times ... \times X_{\alpha_n}) = M_1 \times M_2 \times ... \times F(M_s) \times ... \times M_n$ for $s = 1, 2, ..., n$, then $\overline{F}(\pi_1(\prod_{i=1}^{n} M_i)) = \pi_1(F(\prod_{i=1}^{n} M_i)) \approx \pi_1(M_1) \times \pi_1(M_2) \times ... \times \pi_1(F(M_s)) \times ... \times \pi_1(M_n)$. Also, if $F(\prod_{i=1}^{n} M_i) = F(U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_n}, X_{\alpha_1} \times X_{\alpha_2} \times ... \times X_{\alpha_n}) = (U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_n}, F(U_{\alpha_s}, X_{\alpha_s}), F(U_{\alpha_k}, X_{\alpha_k}), X_{\alpha_1} \times X_{\alpha_2} \times ... \times X_{\alpha_n}) = M_1 \times M_2 \times ... \times F(M_s) \times F(M_k) \times ... \times M_n$ for $s, k = 1, 2, ..., n, s < k$, then $\overline{F}(\pi_1(\prod_{i=1}^{n} M_i)) = \pi_1(F(\prod_{i=1}^{n} M_i)) \approx \pi_1(M_1) \times \pi_1(M_2) \times ... \times \pi_1(F(M_s)) \times \pi_1(F(M_k)) \times ... \times \pi_1(M_n)$. Moreover, by continuing this process if $F(\prod_{i=1}^{n} M_i) = F(U_{\alpha_1} \times U_{\alpha_2} \times ... \times U_{\alpha_n}, X_{\alpha_1} \times X_{\alpha_2} \times ... \times X_{\alpha_n}) = (F(U_{\alpha_1}, X_{\alpha_1}), F(U_{\alpha_2}, X_{\alpha_2}), ..., F(U_{\alpha_n}, X_{\alpha_n})) = F(M_1) \times F(M_2) \times ... \times F(M_n)$, then,

$$\overline{F}(\pi_1(\prod_{i=1}^{n} M_i)) = \pi_1(F(\prod_{i=1}^{n} M_i)) \approx \pi_1(F(M_1)) \times ... \times \pi_1(F(M_n)) = \pi_1(\prod_{i=1}^{n} F(M_i)).$$

□

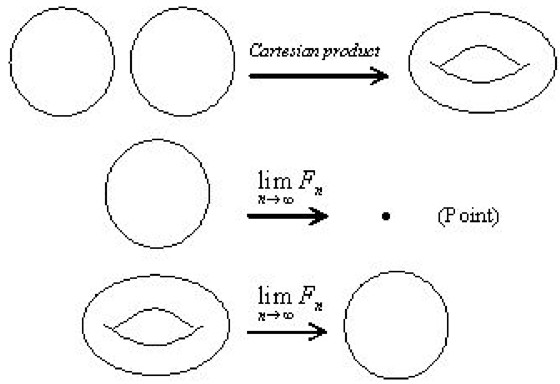
Theorem 5. *If $M_1, M_2, ..., M_n$ are path connected manifolds and F is a folding such that $F(\prod_{i=1}^{n} M_i) \neq \prod_{i=1}^{n} F(M_i)$ then $\pi_1(\lim_{m\to\infty}(F_m(\prod_{i=1}^{n} M_i)) \neq \prod_{i=1}^{n}(\pi_1(\lim_{m\to\infty}(F_m(M_i)))$.*

Proof. Let T^1 be a torus. Take $M_1 = S^1, M_2 = S^1$, then $S^1 \times S^1 = T^1$, since $\lim_{m\to\infty}(F_m(S^1)$ =a point as in Figure 3, then $\lim_{m\to\infty}(F_m(S^1) \times \lim_{m\to\infty}(F_m(S^1) = point$ and so $\pi_1(\lim_{m\to\infty}(F_m(S^1) \times \lim_{m\to\infty}(F_m(S^1)) = 0$.

Also, it follows from $F(S^1 \times S^1) \neq F(S^1) \times F(S^1)$ that $F(S^1 \times S^1) = F(S^1) \times S^1$ or $F(S^1 \times S^1) = S^1 \times F(S^1)$ thus, $\lim_{m\to\infty} F_m(S^1 \times S^1) = S^1$.

So, $\pi_1(\lim_{m\to\infty} F_m(S^1 \times S^1)) = \pi_1(S^1) \approx Z$.

Hence $\pi_1(\lim_{m\to\infty}(F_m(S^1 \times S^1)) \neq \pi_1(\lim_{m\to\infty}(F_m(S^1)) \times \pi_1(\lim_{m\to\infty}(F_m(S^1))$.



□

Corollary 1. *If* $M_1, M_2, ..., M_n$ *are path connected manifolds and* F *is a folding such that* $F(\prod_{i=1}^{n} M_i) = \prod_{i=1}^{n} F(M_i)$ *then* $\pi_1(\lim_{m\to\infty} F_m(\prod_{i=1}^{n} M_i)) \approx \prod_{i=1}^{n}(\lim_{m\to\infty} \pi_1(F_m(M_i)))$.

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