A synthetic approach to the taxicab circles

H. Barış Çolakoğlu - Rüstem Kaya

Abstract. In Euclidean geometry, the following are well known. (1) Through any two distinct points in $\mathbb{R}^2$, infinitely many circles can be constructed. (2) No circle can be constructed through three distinct collinear points in $\mathbb{R}^2$. (3) Through any three distinct non-collinear points in $\mathbb{R}^2$, one and only one circle can be constructed. In [13], Tian-So-Chen discussed the validity of these statements in the taxicab plane geometry with an analytical approach. In this paper, we study, synthetically, the same subject and determine the number of the taxicab circles through two or three distinct points using the concept of bisector.

M.S.C. 2000: 51K05, 51K99.
Key words: taxicab distance, taxicab circle, taxicab bisector, minimum distance set.

§1. Introduction

A family of metrics including the taxicab metric was published in [8] by Minkowski at the beginning of the last century. Later, Menger introduced the taxicab plane geometry in [7] by using the metric $d_T(P_1, P_2)=|x_1 - x_2|+|y_1 - y_2|$ instead of the well known Euclidean metric $d_E(P_1, P_2)=\left((x_1 - x_2)^2+(y_1 - y_2)^2\right)^{1/2}$ for the distance between any two points $P_1=(x_1, y_1)$ and $P_2=(x_2, y_2)$ in the analytical plane. That is, the taxicab distance between the points P_1 and P_2 is the length of a shortest path from P_1 to P_2 composed of the line segments parallel to the coordinate axes. In [5], Krause developed the taxicab geometry, and this geometry has been studied by many authors, see ([1], [2], [3], [4], [6], [9], [10], [11]).

Almost all of the investigations on the taxicab geometry have been made with analytical approach. However, the taxicab geometry is also suitable to study synthetically as Euclidean geometry. Here, we study the problem of existence or nonexistence of the taxicab circles through two or three distinct points with a synthetic approach.

§2. Preliminaries

Definition 1. Let C be a point in the taxicab plane, and r be a positive real number. The set of points

$$\{P \,|\, d_T(C, P) = r\}$$

Applied Sciences, Vol.9, 2007, pp. 67-77.

is called **taxicab circle**, the point C is called **center** of the taxicab circle, and the positive real number r is called **the length of the radius** or simply **radius** of the taxicab circle.

Every taxicab circle in the taxicab plane is an Euclidean square having sides with slope ± 1 as in Figure 1.

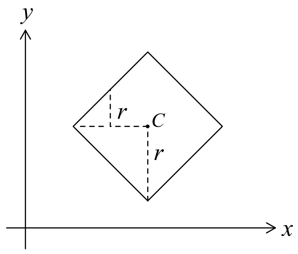


Figure 1

Definition 2. Let l be a line with slope m in the taxicab plane. l is called a **gradual line**, a **steep line** or a **separator** if $|m| < 1$, $|m| > 1$ or $|m| = 1$, respectively (see Figure 2). In particularly, a gradual line is called **horizontal** if it is parallel to x-axis, and a steep line is called **vertical** if it is parallel to y-axis.

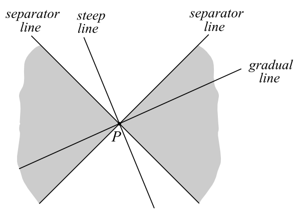


Figure 2

Definition 3. Let A and B be any two distinct points in the taxicab plane. The set of points

$$\{P \,| d_T(A,P) + d_T(B,P) = d_T(A,B)\}$$

is called **minimum distance set** of A and B.

Let A and B be any two distinct points in the taxicab plane, and l denote the line through A and B. It is not difficult to see that

(a) if l is a horizontal or a vertical line, then the minimum distance set of A and B is the line segment AB,
(b) if l is neither a horizontal nor a vertical line, then the minimum distance set of A and B is a rectangular region with diagonal AB, such that each of its sides is parallel to a coordinate axis, as seen in Figure 3.

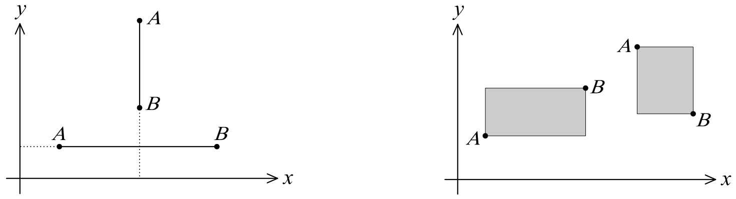


Figure 3

Definition 4. Let A and B be any distinct two points in the taxicab plane. The set of points

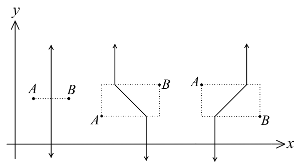
$$\{P d_T(A, P) = d_T(B, P)\}$$

is called **bisector** of A and B.

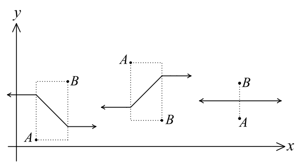
In [12], So and Al-Maskari studied the bisectors and classified them in four categories according to their types. Here, we reclassify them in three categories according to the positions of A and B as follows:

Definition 5. Let l denote the line through any two distinct points A and B. Clearly, l is a gradual line or a steep line or a separator. The bisector of A and B is called a **vertical bisector**, a **horizontal bisector** or a **regional bisector** if l is a gradual line, a steep line or a separator, respectively. Besides, the line segment with slope $+1$ or -1 of a bisector is called **generator** of the bisector.

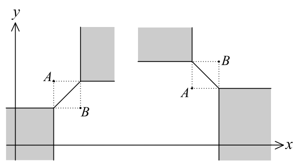
The following figures show us the bisector categories and the bisector types in each categories.



Vertical bisectors
Figure 4



Horizontal bisectors
Figure 5



Regional bisectors
Figure 6

After the definitions above one can easily obtain the following results concerning the number of the points of intersection of two same or different kinds of bisectors:

(2.1) The number of the points of intersection of two vertical bisectors is 0 or 1 or 2 or infinite.

(2.2) The number of the points of intersection of two horizontal bisectors is 0 or 1 or 2 or infinite.

(2.3) The number of the points of intersection of a horizontal and a vertical bisector is 1 or infinite. There are infinitely many points in the intersection if and only if the generators of the bisectors have at least the same two distinct points.

(2.4) The number of the points of intersection of a horizontal and a regional bisector is 1 or infinite.

(2.5) The number of the points of intersection of a vertical and a regional bisector is 1 or infinite.

(2.6) The number of the points of intersection of two regional bisectors is 1 or infinite. There is only one point in the intersection if and only if the generators of the bisectors intersect in exactly one point which is not a common endpoint of them.

§3. Some Lemmas About Intersection of Bisectors

Now, we give some lemmas leading to the main theorem which gives a complete answer to the problem. In what follows, "a side of a line l" means "the union of the line l and a half-plane with edge l". That is, both side of a line l contains the line l.

Lemma 1. *Bisectors of two pairs of points, each of which lies on a gradual line, do not have any point of intersection if the minimum distance sets of these two pairs of points lie in opposite sides of a vertical line.*

Proof. Let A, B and C, D be two pairs of points, each of which lies on a gradual line in the taxicab plane. Then the bisectors of A, B and C, D are vertical. If the minimum distance sets of A, B and C, D lie in opposite sides of a vertical line v, then the bisectors of the pairs of points lie in opposite sides of v, and do not intersect (see Figure 7). Thus, these bisectors do not have any point of intersection. □

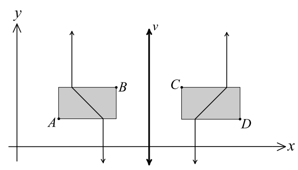


Figure 7

Lemma 2. *Bisectors of two pairs of points, each of which lies on a steep line, do not have any point of intersection if the minimum distance sets of these two pairs of points lie in opposite sides of a horizontal line.*

Proof. Proof is similar to that of Lemma 6. □

Lemma 3. *Bisectors of two pairs of points, each of which lies on a separator, have infinitely many points of intersection if the minimum distance sets of these two pairs of points lie in opposite sides of a vertical or a horizontal line.*

Proof. Let A, B and C, D be two pairs of points, each of which lies on a separator in the taxicab plane. Then the bisectors of A, B and C, D are regional. If the minimum distance sets of A, B and C, D lie in opposite sides of a vertical line v

or a horizontal line h, then the generators of the bisectors of A, B and C, D either intersect in a common endpoint of the generators or do not intersect (see Figures 8.a,b,c,d). Thus, these bisectors have infinitely many points of intersection. $\square$

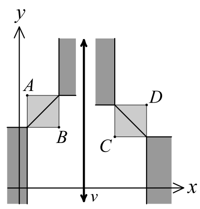


Figure 8.a

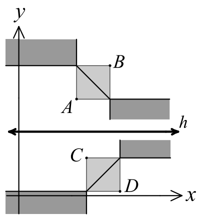


Figure 8.b

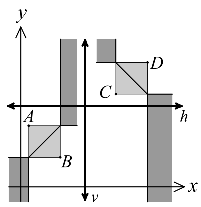


Figure 8.c

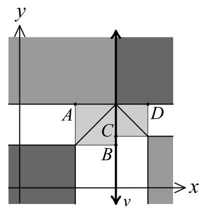


Figure 8.d

Lemma 4. *Bisectors of two pairs of points, one of which lies on a separator and the other one lies on a gradual line, have infinitely many points of intersection if the minimum distance sets of these two pairs of points lie in opposite sides of a vertical line.*

Proof. Let A, B be a pair of points on a separator, and C, D be a pair of points on a gradual line in the taxicab plane. Then the bisector of A, B is regional, and the bisector of C, D is vertical. If the minimum distance sets of A, B and C, D lie in opposite sides of a vertical line v, then a vertical ray of the vertical bisector and a planar region of the regional bisector have infinitely many points of intersection (see Figure 9). Thus, these bisectors have infinitely many points of intersection. $\square$

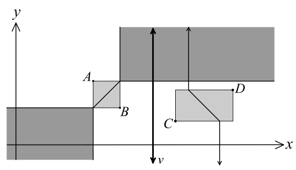


Figure 9

Lemma 5. *Bisectors of two pairs of points, one of which lies on a separator and the other one lies on a steep line, have infinitely many points of intersection if the minimum distance sets of these two pairs of points lie in opposite sides of a horizontal line.*

Proof. Proof is similar to that of Lemma 9. $\square$

Lemma 6. *Bisectors of two pairs of points, one of which lies on a gradual line and the other one lies on a steep line, have exactly one point of intersection if the generators of the bisectors lie in the same separator.*

Proof. Let A, B be a pair of points on a gradual line, and C, D be a pair of points on a steep line in the taxicab plane. Then the bisector of A, B is vertical, and the bisector of C, D is horizontal. If the generators of the bisectors of A, B and C, D do not lie in the same separator, then the generators have either exactly one or no points of intersection. Since a vertical and a horizontal bisectors have infinitely many points of intersection if and only if their generators have infinitely many points of intersection, the bisectors of A, B and C, D have exactly one point of intersection (see Figures 10.a,b). □

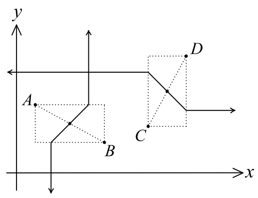


Figure 10.a

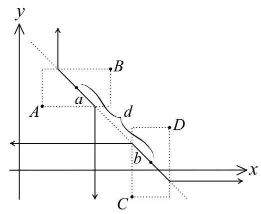


Figure 10.b

Lemma 7. *Bisectors of two pairs of points, one of which lies on a gradual line and the other one lies on a steep line, (i) have infinitely many points of intersection if the generators of the bisectors lie in the same separator and the sum of the lengths of the generators is bigger than two times of the distance between midpoints of the pairs of points, (ii) have exactly one point of intersection if the generators of the bisectors lie in the same separator and the sum of the lengths of the generators is not bigger than two times of the distance between midpoints of the pairs of points.*

Proof. Let A, B be a pair of points on a gradual line, and C, D be a pair of points on a steep line in the taxicab plane. Then the bisector of A, B is vertical, and the bisector of C, D is horizontal. Let the generators of the bisectors of A, B and C, D lie in the same separator, the lengths of the generators be $2a$ and $2b$, and the distance between midpoints of A, B and C, D be d. If $(2a + 2b) > 2d$, then $(a + b) > d$, and the bisectors have infinitely many points of intersection (see Figure 11.a). If $(2a + 2b) \leq 2d$, then $(a + b) \leq d$, and the bisectors have exactly one point of intersection (see Figure 11.b). □

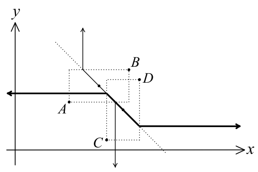


Figure 11.a

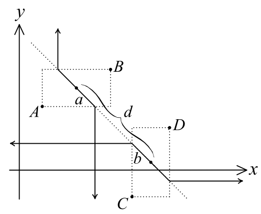


Figure 11.b

§4. Main Theorem

The following theorem shows that statement (1) given in the abstract is valid but the statements (2) and (3) are invalid in the taxicab plane, and determines in what conditions how many circles can be constructed through given distinct two or three points in the taxicab plane.

Theorem. *In the taxicab plane, the following statements are valid:*
(4.1) Through any two distinct points, infinitely many taxicab circles can be constructed.
(4.2) No taxicab circle can be constructed through three distinct collinear points lying on a gradual line.

(4.3) No taxicab circle can be constructed through three distinct collinear points lying on a steep line.
(4.4) Infinitely many taxicab circles can be constructed through three distinct collinear points lying on a separator.
(4.5) No taxicab circle can be constructed through any three distinct non-collinear points which determine three gradual lines.
(4.6) No taxicab circle can be constructed through any three distinct non-collinear points which determine three steep lines.
(4.7) Infinitely many taxicab circles can be constructed through three distinct non-collinear points which determine a separator and two gradual lines.
(4.8) Infinitely many taxicab circles can be constructed through three distinct non-collinear points which determine a separator and two steep lines.
(4.9) Infinitely many taxicab circles can be constructed through three distinct non-collinear points which determine two separators and a gradual line.
(4.10) Infinitely many taxicab circles can be constructed through three distinct non-collinear points which determine two separators and a steep line.
(4.11) One and only one taxicab circle can be constructed through any three distinct non-collinear points which determine a gradual line and two steep lines.
(4.12) One and only one taxicab circle can be constructed through any three distinct non-collinear points which determine a steep line and two gradual lines.
(4.13) Through three distinct non-collinear points which determine a separator, a gradual line and a steep line, (i) infinitely many taxicab circles can be constructed if the point which is not on the separator lies out of the circles, whose vertices are the remaining two points determining the separator, (ii) one and only one taxicab circle can be constructed if the point which is not on the separator do not lie out of the circles, whose vertices are the remaining two points determining the separator.

Proof. Clearly, the number of the circles through three distinct points is equal the number of the points of intersection of the bisectors of any two pairs of points from these points. This property has a main role in the proof of the theorem.

(4.1) It is obvious that there exist infinitely many points equidistant from any two distinct points in the taxicab plane. Therefore, through any two distinct points, infinitely many circles can be constructed.

(4.2) Let A, B and C be three distinct collinear points lying on a gradual line, such that B is between A and C in the taxicab plane (see Figure 12). Then the bisectors of A, B and B, C are vertical, and the minimum distance sets of A, B and B, C lie in opposite sides of the vertical line v through B. From Lemma 1, these bisectors do not have any point of intersection. Thus, no taxicab circle can be constructed through three distinct collinear points lying on a gradual line.

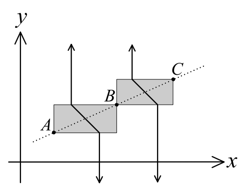


Figure 12

(4.3) Proof is similar to that of (4.2), using Lemma 2 instead of Lemma 1.

(4.4) Let A, B and C be three distinct collinear points lying on a separator, such that B is between A and C in the taxicab plane (see Figure 13). Then the bisectors of A, B and B, C are regional, and the minimum distance sets of A, B and B, C lie in opposite sides of the vertical line v through B, or horizontal line h through B. From Lemma 3, these bisectors have infinitely many points of intersection. Thus, infinitely many taxicab circles can be constructed through three distinct collinear points lying on a separator.

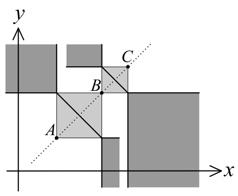


Figure 13

(4.5) Let A, B and C be three distinct non-collinear points which determine three gradual lines, such that B is between A and C according to their x-coordinates (see Figure 14). Then the bisectors of A, B and B, C are vertical, and the minimum distance sets of A, B and B, C lie in opposite sides of the vertical line v through B. From Lemma 1, these bisectors do not have any point of intersection. Thus, no taxicab circle can be constructed through three distinct non-collinear points which determine three gradual lines.

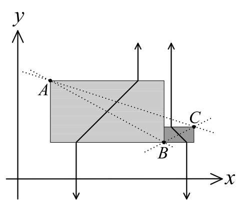


Figure 14

(4.6) Proof is similar to that of (4.5), using Lemma 2 instead of Lemma 1.

(4.7) Let A, B and C be three distinct non-collinear points which determine a separator and two gradual lines, such that A and B determine the separator. Then the bisector of A, B is regional, and the bisectors of A, C and B, C are vertical, and either A is between B and C or B is between A and C according to their x-coordinates. If B is between A and C, then the minimum distance sets of A, B and B, C lie in opposite sides of the vertical line v through B (see Figure 15.a). If A is between B and C, then the minimum distance sets of A, B and A, C lie in opposite sides of the vertical line v' through A (see Figure 15.b). From Lemma 4, these bisectors have infinitely many points of intersection. Thus, infinitely many taxicab circles can be constructed through three distinct non-collinear points which determine a separator and two gradual lines.

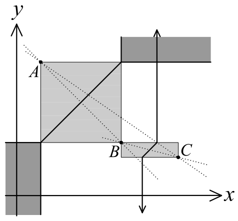


Figure 15.a

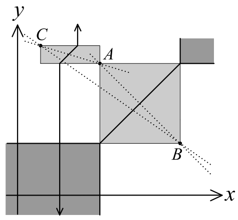


Figure 15.b

(4.8) Proof is similar to that of (4.7), using Lemma 5 instead of Lemma 4.

(4.9) Let A, B and C be three distinct non-collinear points which determine two separators and a gradual line, such that A and C determine the gradual line. Then the bisectors of A, B and B, C are regional, and B is between A and C according to their x-coordinates, and the minimum distance sets of these pairs of points lie in opposite sides of the vertical line v through B (see Figure 16). From Lemma 3, these bisectors have infinitely many points of intersection. Thus, infinitely many taxicab circles can be constructed through three distinct non-collinear points which determine two separators and a gradual line.

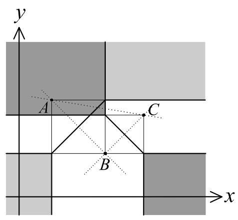


Figure 16

(4.10) Proof is similar to that of (4.9).

(4.11) Let A, B and C be three distinct non-collinear points which determine a gradual and two steep lines, such that A and B determine the gradual line. Then the bisector of A, B is vertical, and the bisector of A, C is horizontal, and the midpoints of A, B and A, C determine a steep line. Since the midpoints of A, B and A, C are not on the same separator, the generators of the bisectors of A, B and C, D do not lie in the same separator (see Figure 17). From Lemma 6, these bisectors have exactly one point of intersection. Thus, one and only one taxicab circle can be constructed through any three distinct non-collinear points which determine a gradual line and two steep lines.

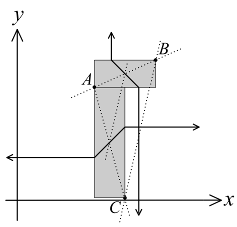


Figure 17

(4.12) Proof is similar to that of (4.11).

(4.13) Let A, B and C be three distinct non-collinear points which determine a separator, a gradual line and a steep line, such that A and B determine the separator. Then C lies in the shaded region in Figure 18.a, and the one of the bisectors of A, C and B, C is a vertical, and the other one is horizontal.

(i) If C lies out of the circles with vertices A and B, then the generators of the bisectors of A, C and B, C lie in the same separator, and the sum of the lengths of the generators is bigger than two times of the distance between midpoints of A, C and B, C (see Figure 18.b). From Lemma 7, these bisectors have infinitely many points of intersection. Thus, infinitely many taxicab circles can be constructed through A, B and C.

(ii) If C does not lie out of the circles with vertices A and B, then either the generators of the bisectors of A, C and B, C do not lie in the same separator or the generators of the bisectors of A, C and B, C lie in the same separator and the sum of the lengths of the generators is not bigger than two times of the distance between midpoints of A, C and B, C (see Figures 18.c,d). From Lemma 6 and Lemma 7, the bisectors have exactly one point of intersection. Thus, only one taxicab circle can be constructed through A, B and C. □

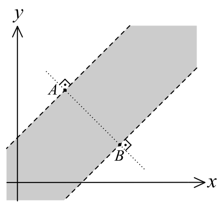


Figure 18.a

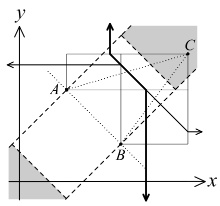


Figure 18.b

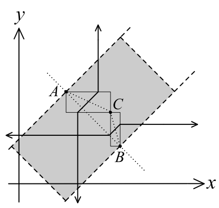


Figure 18.c

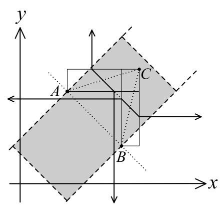


Figure 18.d

References

[1] O. Gelisgen, R. Kaya, *On α-distance in three dimensional space*, Applied Sciences (APPS), 8 (2006), 65-69.

[2] Y. P. Ho, Y. Liu, *Parabolas in Taxicab Geometry*, Missouri J. of Math. Sci., 8 (1996), 63-72.

[3] R. Kaya, Z. Akca, I. Günaltili, M. Özcan, *General equation for taxicab conics and their classification*, Mitt. Math. Ges Hamburg, 19 (2000), 135-148.

[4] R. Kaya, H. B. Colakoglu, *Taxicab versions of some Euclidean theorems*, Int. Jour. of Pure and Appl. Math. (IJPAM) 26, 1 (2006), 69-81.

[5] E.F. Krause, *Taxicab Geometry*, Addision-Wesley, 1975.

[6] R. Laatsch, *Pyramidal sections in Taxicab Geometry*, Mitt. Mathematics Magazine, 55 (1982), 205-212.

[7] K. Menger, *You Will Like Geometry, Guildbook of the Illinois Institute of Technology Geometry Exhibit*, Museum of Science and Industry, Chicago, IL, 1952.

[8] H. Minkowski, *Gesammelte Abhandlungen*, Chelsa Publishing Co. New York, 1967.

[9] M. Ozcan, R. Kaya, *On the ratio of directed lengths in the taxicab plane and related properties*, Missouri J. of Math. Sci., 14 (2002), 107-117.

[10] B. E. Reynolds, *Taxicab Geometry*, Pi Mu Epsilon Journal, 7 (1980), 77-78.

[11] D. J. Schattschneider, *The taxicab group*, Amer. Math. Monthly, 91 (1984), 423-428.

[12] S. S. So, Z. S. Al-Maskari, *Two Simple Examples in Non-Euclidean Geometry*, Kansas Science Teacher (J. of Math. and Sciece Teaching), 11 (1995), 14-18.

[13] S. Tian, S. S. So, G. Chen, *Concerning circles in Taxicab Geometry*, J. Math. Educ. Sci. Technol. 28 (1997), 727-733.

Authors' addresses:

Harun Barış Çolakoğlu and Rüstem Kaya
Eskişehir Osmangazi University, Faculty of Arts and Sciences,
Department of Mathematics, 26480, Eskişehir, Turkey.
e-mail : bariscolakoglu@hotmail.com, rkaya@ogu.edu.tr