Dynamical analysis of a demand governed market

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Abstract. We use a modified Rosenzweig-Mac Arthur ODE system in order to model a market in which innovative products are launched according to a growth function influenced by both producers and consumers. Interesting properties of the related dynamics, such as the uniqueness of a limit cycle, can easily be proved.

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§1. Introduction

The predator-prey dynamics have often been used for in-depth analysis of the behaviour of interactive economic agents in various kinds of markets. The classical literature is vast: consider, for instance, the Goodwin-style models (e.g. Goodwin, [5]) of economic growth, where workers' participation in revenue is preyed upon the rate of employees.

In a previous paper [8] we exhaustively examined the competitive dynamics arising and consolidating in and between industrial sectors, and offered an analysis which featured a few characteristic elements of predator-prey models (see Volterra, [10, 11]; Kolmogoroff, [7]). We especially referred to behaviours in the most innovative technological sectors in terms of R & D, consequently to the whole field of introduction and distribution of communication and information technologies.

We believe that the dynamic evolution of these markets deserves special attention, not only because of the specific characteristics of the supply functions but also for some peculiarities characterizing the patterns of demand. Indeed, the demand conditions (preferences, size of user base, state of knowledge, etc.) have to be adapted as they change over time, following the natural course of the product or service, from its gradual introduction into a market to its possible mass-production and/or subsequent abandonment.

This particular category of goods and services is endowed with different potentialities depending on the way they are used: those products are able to satisfy needs when used individually, but they even increase their utility by means of connection and integration with other commodities in the same industrial sector. For instance, consider the extremely fast development since the launch of the first personal computer: in the latest years, its sales have been exceeding those of the "historical" media

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appliance, such as television, in nearly all European countries (after the US). Nowadays another boom in the PC sector, due to the integrative capacities provided by the diffusion of TCP/IP protocols, is grandly spreading out all over the world. Those protocols were first created independently as connectivity instruments responding to military (and subsequently academic) needs, then they became the framework of the Internet phenomenon with which today we all deal. In order to exploit the potential of the World Wide Web and to make it accessible to as many people as possible, the technologies of hardware and software have been developing fruitful interdependencies and synergies. Consequently, this instrument has been able to give a huge number of users the chance to construct relationships and to create opportunities for work and diffusion of knowledge in every field, hence to develop furthermore.

An analogous explosion has been taking place as far as the mobile phones are concerned: in the beginning they were looked upon as an useless luxury few people could afford. Subsequently, after being enriched by new technologic functions and potentialities, they have been becoming an inalienable instrument for both work and free time. Besides, the history of their evolution follows the classical path of technologic goods: every time a new standard is born (GSM, UMTS, etc.) many new consumers leave the past technology to follow the present one, which spreads immediately while the old one gets progressively abandoned and decays.

In this paper we shall base our analysis on an ODE model to examine the interactions between a certain number of potential consumers $u(t)$ of a given good and a certain amount of businesses $v(t)$, dealing with the same technology for commodity production. Our aim is to provide an interpretation of the manufacturer(predator)-consumer(prey) dynamics by making use of a model similar to the well-known one of Rosenzweig-Mac Arthur, see [9], whose equations show as follows:

$$\begin{cases} u'(t) = u(t)(f(u(t)) - v(t)) \\ v'(t) = v(t)(u(t) - \sigma) \end{cases} \tag{1.1}$$

Widely studied in Biomathematics, these equations feature a bifurcation parameter σ and a demand function representing the market growth rate $f(u)$.

Let us briefly recall the traditional hypotheses for $f(u)$ as enunciated in [4]: it should be a strictly concave $C^2([0, +\infty), \mathbb{R})$ function, having only two positive zeros b_1 and b_2 and reaching its maximum in $m \in (b_1, b_2)$. There exists a meaningful difference between this layout and the one proposed in [8], where the hypothesis $f(0) > 0$ expressed a market situation that reacted to product disappearance with hoarding phenomena. The model we intend to analyze consists in a simple but essential modification of the Rosenzweig-Mac Arthur equations: we are going to replace the growth function $f(u)$ with a two-variable map $F(u, v)$, depending on the evolution of purchasers as well as on the manufacturers'one. By choosing a function of the kind $F(u, v) = f(u) \cdot \phi(v)$, it will be not difficult for us to preserve the original hypotheses on the previous growth dynamics if $\phi(v)$, the contributions of the businesses to the growth rate, is positive for all $v \geq 0$.

Consequently, the dynamic system we shall take into examination will have the following form:

$$\begin{cases} u'(t) = u(t)(f(u(t))\phi(v(t)) - v(t)) \\ v'(t) = v(t)(u(t) - \sigma) \end{cases} \tag{1.2}$$

where we assume that:

a) $f(0) < 0$;

b) $\phi(v) > 0$ for all $v \geq 0$;

c) there exist $b_1,\ b_2 \in (0, +\infty)$, $b_1 < b_2$ such that $f(b_1) = f(b_2) = 0$, and $m \in (b_1, b_2)$ such that $f'(u) > 0$ if $u \in [b_1, m)$ and $f'(u) < 0$ if $u \in (m, b_2]$;

d) $f \in C^2([0, +\infty), \mathbb{R})$, admits third derivative in m, and $f''(u) < 0$ for all $u \in [0, +\infty)$;

e) $b_1 < \sigma < b_2$.

Our goal is the description of the bifurcations in system (1.2) as the parameter σ changes in its interval. Especially referring to the IT market, the economic meaning of the hypotheses can be summarized as follows:

a) represents an initial market contraction, in order to transpose the ecological notion of asocial prey into economics. On the other hand, when the number of consumers is close to zero, the evolutionary consumption rate is negative, and that expresses two realistic facts. First, the absence of a given technology may push the market to turn its attention to goods created using alternative technologies. Besides, it may not be easy for innovative technologies to penetrate consolidated markets.

b) The contribution provided by the producers to the market growth rate has always to be thought of as positive, because of the natural fact that all efforts made by businessmen aim at increasing the consumption rate.

c) Since we are not considering a demand that can expand infinitely, we hypothesize that its growth rates take on the traditional logistic curve pattern: the initial market contraction, and the consequential recession period, stop when the number of consumers reaches a certain threshold value b_1. Subsequently, the growth curve reaches a peak, and then declines until it intersects the u axis in b_2, the second threshold value. The smaller value can be interpreted as the initial critical mass below which the number of consumers begin to contract in relation to the placement of commodities deriving from the technology in question (just like the extinction of a species in ecological models). On the other hand, the bigger value represents the market threshold level, i.e. the maximum amount of possible users for that specific technology. For values which are bigger than b_2, the sign of the market growth function changes, and this symbolizes the saturation of the market. From our viewpoint, a single reversal in the demand pattern is sufficient to provide a suitable portrait of the market in the short/medium term situations we intend to analyze. If we theorized more intervals in which the monotonicity of $f(u)$ changed, indicating several stages of expansion and contraction of the market, such a hypothesis would be allowed to invalidate the condition of coeteris paribus in the other data, especially the prices in relation to the remaining goods.

d) This is a solely technical hypothesis aimed at ensuring proper calculation of the bifurcation direction.

e) The constraint on the bifurcation parameter is supposed to prevent the paradox that manufacturing technology might implode for values of $u(t)$ close to the market threshold level.

As we shall see in next Section, when the businesses' contribution to the growth rate is constant, the analysis practically yields the well-known results by Conway and Smoller [4]. On the other hand, by adding some hypotheses to the growth function, we shall be able to prove some peculiar properties. In particular, if the consumers' contribution function has a certain symmetry and the businesses' contribution function is nonincreasing and limited from below, a limit cycle for the system exists and it is unique.

§2. Modified Rosenzweig-Mac Arthur equations in autonomous form

In the hypotheses we formulated, (1.2) is actually a predator-prey model: the first quadrant of the phase plane is an invariant set for (1.2). Before beginning a deep analysis of the qualitative properties of the system, we should briefly focus how a simple choice of the demand function recreates the classical Rosenzweig-Mac Arthur model, and outline some old results about it.

2.1 The Conway-Smoller analysis

First, let us consider the case $F(u, v) = kf(u)$, where $\phi(v) = k > 0$, in which the contribution of businesses to the market growth rate is constant. The analysis we are lead to is essentially equal to the classical one exposed by Conway and Smoller (1986), with respect to which almost nothing changes. It is sensible to recall some peculiar aspects of that system:

$$\begin{cases} u' = u(kf(u) - v) \\ v' = v(u - \sigma) \end{cases}$$

The four equilibrium points of the system are: $0 \equiv (0; 0)$, $B_1 \equiv (b_1; 0)$, $B_2 \equiv (b_2; 0)$, $E \equiv (\sigma, kf(\sigma))$. We shall study the linearization at equilibria by investigating the spectrum of the Jacobian matrix:

$$J(u, v) = \begin{pmatrix} kf(u) - v + kuf'(u) & -u \\ & \\ v & u - \sigma \end{pmatrix}$$

hence it is straightforward to notice that:

1. O is a stable node, whose invariant manifolds are the coordinate axes u and v; in fact, $J(0; 0)$ has the negative eigenvalues $\lambda_1 = kf(0)$, $\lambda_2 = -\sigma$. The economic significance of the configuration is a sort of market pit: the simultaneous disappearance of a certain good or its potential purchasers obviously causes the product itself to disappear: both the market and technologies turn their attention elsewhere.

2. B_1 is a saddle point, because the eigenvalues of the matrix $J(b_1; 0)$ are $\lambda_1 = kb_1 f'(b_1) > 0$, $\lambda_2 = b_1 - \sigma < 0$, consequently the unstable manifold $M_\sigma^U(b_1; 0)$ coincides with the u axis and the stable manifold $M_\sigma^S(b_1; 0)$ enters the B_1 equilibrium above the curve $v = f(u)$.

3. B_2 is another saddle point, since the eigenvalues of $J(b_2; 0)$ are

$$\lambda_1 = kb_2 f'(b_2) < 0, \ \lambda_2 = b_2 - \sigma > 0,$$

with stable manifold on the u axis and unstable manifold $M_\sigma^U(b_2; 0)$ which exits from B_2 and enters the region: $\{(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid v > f(u)\}$.

4. E requires a Hopf bifurcation analysis. The eigenvalues of the variational matrix evaluated in E are:

$$\lambda_{1,2} = \frac{k\sigma f'(\sigma) \pm \sqrt{k\sigma \left[k\sigma \left(f'(\sigma)\right)^2 - 4f(\sigma)\right]}}{2},$$

so E is a stable node if $m < \sigma < b_2$, an unstable node if $b_1 < \sigma < m$, a center if $\sigma = m$.

The economic meaning of the fixed points is that both in the instance where σ assumes values close to the first zero b_1 of the evolutionary function of the market $f(u)$, and where σ is close to the second zero b_2, there is no insurgence of economic cycle phenomena. In the first case, the (unstable) node saddle connection represents the extinction of the product's manufacturers. In the second instance, the saddle (stable) node connection, production survives and settles around a constant level of manufacturers for the product, which can therefore enter the market. In this case we have the typical situation of a market with multiple equilibria: depending on the situation, the market may settle around a low level of equilibrium, in which product penetration is limited and close to the level b_1. However, this situation is unstable, and a small positive perturbation can set off an disequilibrium dynamic leading to the next high level, where the product's penetration and diffusion reach a larger share of consumers. This situation is stable, and thus able to maintain itself over time even against local perturbations in conditions.

But the most interesting characteristics of the system are those that refer to the possible presence of cyclical phenomena, due to the characteristics of the demand function. This situation is linked to the possible existence of a Hopf bifurcation in the transition from the stable to the unstable area. In fact consider the Hopf bifurcations in E for $\sigma = m$. The coefficient matrix of the linearization of (1.1) in this case has eigenvalues $\lambda_{1,2} = \pm i\sqrt{mf(m)}$, consequently there is a Hopf bifurcation in $\sigma = m$. The direction of the bifurcation is thus determined by the quantity:

$$\mu = -\frac{2}{m} - \frac{f'''(m)}{f''(m)}.$$

It is here that the technical hypothesis (d) on the regularity of f is used. More specifically followine [6], we may state the following:

Theorem 2.1. *If $m < 0$ there is a locally unique bifurcation solution leaving E with a stable and periodic orbit as soon as σ falls below m; thus the periodic solution is defined for fairly small values of $\sigma < m$ and $|\sigma - m|$*

If $m > 0$ there is a locally unique bifurcation solution leaving E with an unstable and periodic orbit as soon as σ rises above m.

Proof. The thesis follows from the Hopf bifurcation theorem and the results on the stability direction, see [6]. □

Note that the two possibilities cited by Theorem 2.1 may occur for the system (1.1) in question. If we consider the case of a quadratic function of consumption evolution $f(u) = -a(u - b_2)(u - b_1)$, we find that $\mu = -2/m < 0$; while if $f(u) = a(1 - u/c_2) - c_1 e^{-bu}$ then $\mu > 0$ if and only if $\ln(c_1 b c_2/a) > 2$, while $\mu < 0$ if and only if $\ln(c_1 b c_2/a) < 2$.

In the special case of a symmetrical market evolution function in relation to the line $u = m$, it is possible to extend a result by Cheng [2], in order to obtain the unique periodic solution of (1.1). In this case, the system produces a dynamic well known to economic cycle scholars: if there is a possibility for multiple equilibria that identify alternately stable and unstable regions, the transition from one region to another may generate periodic orbits in the presence of certain parameter values. These cycles are in turn only partially stable (depending on the situation, only endogenously or only exogenously). The market may therefore prompt an especially interesting cyclical dynamic, indicating the curious situation of variable growth rates over time, conditions which are relatively uncertain for businesses. One may therefore wonder: how do businesses react to this type of situation? The existence of a Hopf phenomenon is closely related to the value of the parameter σ, a situation that may be controlled by businesses in the sector, and probably changed by the same parameters of R&D and competitiveness that define such a highly innovative industry.

Theorem 2.2. *If f is symmetrical with respect to the line $u = m$ then every periodic orbit of (1.1) is a stable-orbit attractor, and thus only one of such orbits may exist.*

Proof. See [4]. □

Let us stress that the case of f symmetrical is typical if the market growth rate is Malthusian, i.e.

$$f(u) = -a(u - b_2)(u - b_1).$$

2.2 Different growth dynamics

If we replace the hypothesis b) with the following one:

b') $\phi(v) \geq 0$ for all $v \geq 0$,

we can consider a linear contribution of the manufacturers to the growth function of the model, i.e. $\phi(v) = \alpha v$, $\alpha > 0$. This modification seems to make sense since any agent which is close to disappearance is only able to produce negligible effects on the demand growth. On the other hand, this choice would imply a constant increase in the contribution of businesses to the demand function, but this could cause an overestimate phenomenon of the producers' actual power. However, considering the initial conditions as strictly positive to avoid problems in convergence of the integral solution, the dynamic system is immediately integrable by separation of variables. If $u(t_0) = u_0 > 0$, $v(t_0) = v_0 > 0$, the corresponding solution can be written down as follows:

$$\frac{du}{dv} = \frac{u(\alpha f(u) - 1)}{u - \sigma} \Longrightarrow v(u) = v(u_0) + \int_{u_0}^{u} \frac{s - \sigma}{s(\alpha f(s) - 1)} ds.$$

Now, let us consider different forms for $\phi(v)$: the first remarkable novelty concerns the number of equilibria. Obviously, we still have the fixed points of the original model, i.e. $O \equiv (0, 0)$, $B_1 \equiv (b_1, 0)$, $B_2 \equiv (b_2, 0)$, which keep the same behaviour. In fact, the Jacobian matrix $J(u, v)$ has the following structure:

$$J(u, v) = \begin{pmatrix} f(u)\phi(v) - v + uf'(u)\phi(v) & u(f(u)\phi'(v) - 1) \\ v & u - \sigma \end{pmatrix}.$$

$J(0, 0)$ is a diagonal matrix with eigenvalues $f(0)\phi(0)$ and $-\sigma$, both strictly negative, so the origin is a saddle point and the economic meaning of this situation remains the same as it ever was.

B_1 and B_2 are saddle points for (1.2), because for all $\sigma \in (b_1, b_2)$, $\det(J(B_k)) < 0$ for $k = 1, 2$. In addition to these points, the system can admit some more equilibria; if we consider the equation:

$$f(\sigma) = \frac{v}{\phi(v)} := G(v), \tag{2.3}$$

it is straightforward to see that if σ^* satisfies (2.3), the points (σ, σ^*) are stationary points for (1.2). $G(v)$ is a positive continuous function which vanishes in $v = 0$. We can prove that if (2.3) has more than one solution, i.e. the points $P_i \equiv (\sigma, \sigma_i^*)$, $i = 1, \dots, k$, in increasing order, that is $\sigma_i^* < \sigma_j^*$ for $i < j$, the steady states P_i, where i is an even integer, are all saddle points for every value of σ, whereas all the others are stable nodes if $f'(\sigma) < 0$, unstable nodes if $f'(\sigma) > 0$.

Theorem 2.3. *If there exist v_1, $v_2 > 0$ such that*

$$G'(v_1) = G'(v_2) = 0, \text{ and } G(v_1) > f(\sigma) > G(v_2),$$

then there exist $\sigma_1^* \in (0, v_1)$, $\sigma_2^* \in (v_1, v_2)$ such that:

1) $P_1 \equiv (\sigma, \sigma_1^*)$ is an equilibrium point of the system (1.2), specifically a stable node, if $\sigma \in (m, b_2)$, an unstable node if $\sigma \in (b_1, m)$, a center if $\sigma = m$;

2) $P_2 \equiv (\sigma, \sigma_2^*)$ is a saddle point, for every $\sigma \in (b_1, b_2)$.

Proof. Since $G(v)$ is a continuous function, the existence of the two solutions σ_1^* and σ_2^* of (2.3) is obvious by hypothesis. The Jacobian matrix evaluated in P_1 is:

$$J(P_1) = \begin{pmatrix} f(\sigma)\phi(\sigma_1^*) - \sigma_1^* + \sigma f'(\sigma)\phi(\sigma_1^*) & \sigma(f(\sigma)\phi'(\sigma_1^*) - 1) \\ \sigma_1^* & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \sigma f'(\sigma)\dfrac{\sigma_1^*}{f(\sigma)} & \sigma\left(\dfrac{\sigma_1^*\phi'(\sigma_1^*)}{\phi(\sigma_1^*)} - 1\right) \\ \sigma_1^* & 0 \end{pmatrix},$$

because $f(\sigma) = \sigma_1^*/\phi(\sigma_1^*)$. Since v_1 is a local maximum for G, we have that $G'(\sigma_1^*) > 0$, consequently

$$\phi(\sigma_1^*) - \sigma_1^*\phi'(\sigma_1^*) > 0 \implies$$

$$\implies \frac{\sigma_1^*\phi'(\sigma_1^*)}{\phi(\sigma_1^*)} < 1 \implies \det(J(P_1)) > 0.$$

This implies that P_1 is a stable node if and only if $\mathrm{tr}(J(P_1)) < 0$. It is immediate to realize that the nature of P_1 only depends on the sign of $f'(\sigma)$. If $\sigma \in (b_1, m)$, $f'(\sigma) > 0$, and P_1 is unstable. If $\sigma = m$, $f'(\sigma) = 0$, $\mathrm{tr}(J(P_1)) = 0$, and P_1 is a center of a periodic orbit. If $\sigma \in (m, b_2)$, $f'(\sigma) < 0$, and P_1 is stable.

Analogously, we have that:

$$J(P_2) = \begin{pmatrix} \sigma f'(\sigma)\dfrac{\sigma_2^*}{f(\sigma)} & \sigma\left(\dfrac{\sigma_2^*\phi'(\sigma_2^*)}{\phi(\sigma_2^*)} - 1\right) \\ \sigma_2^* & 0 \end{pmatrix},$$

and repeating the previous analysis, since $G'(\sigma_2^*) < 0$, $\det(J(P_2)) < 0$, so P_2 is a saddle point for (2). □

A suitable choice for $\phi(v)$ is not difficult to find: any polynomial $a + bv + cv^2$ where $a,\ b,\ c \in \mathbb{R}_+$ may work. In fact, $G(v) = \dfrac{v}{a + bv + cv^2}$ tends to 0 as $v \longrightarrow +\infty$ and reaches its maximum in the point $v^* = \sqrt{\dfrac{a}{c}}$, i.e. $G(v^*) = \dfrac{1}{2\sqrt{ac} + b}$.

Consequently, if

$$G(v^*) > \sup_{u\in[0,+\infty)} f(u) = f(m),$$

certainly (2.3) has two solutions, and previous proposition can be applied.

Example 2.1. By choosing $f(u) = -(u-1)(u-2)$, that is $b_1 = 1,\ b_2 = 2,\ m = 3/2$, and $\phi(v) = 1 + v/2 + v^2$, the two intersection points we obtain are given by:

$$v_{1,2} = \frac{\sigma^2 - 3\sigma + 4 \pm \sqrt{-15\sigma^4 + 90\sigma^3 - 191\sigma^2 + 168\sigma - 48}}{-4\sigma^2 + 12\sigma - 8}.$$

2.3 Uniqueness of a limit cycle

In this subsection we shall reformulate Theorem 2.2 in order to extend the result to system (1.2), by adapting the classical construction by Cheng, [2], to our system, showing the uniqueness of a limit cycle for system (1.2) under some hypotheses on the function $\phi(v)$.

Theorem 2.4. *If $f(u)$ is symmetrical with respect to the line $u = m$, $\phi(v) \geq 1$ for all $v \geq 0$, and $\phi(v)$ is nonincreasing, then every periodic orbit of (1.2) is a stable-orbit attractor, and thus only one of such trajectories may exist.*

Proof. We shall prove this assertion by applying Cheng's technique to our system, with a few slight differences. Consider the divergence of the vector field $(u', v') = (u(f(u)\phi(v) - v),\ v(u - \sigma))$, i.e.:

$$\mathrm{div}(u', v') = f(u)\phi(v) - v + uf'(u)\phi(v) + u - \sigma;$$

we intend to evaluate it on the circuit Γ, nontrivial closed orbit of system (2). Since

$$\int_\Gamma (u(t) - \sigma)dt = \int_\Gamma \frac{dv}{v} = 0, \quad \text{and} \quad \int_\Gamma (f(u(t))\phi(v(t)) - v(t))dt = \int_\Gamma \frac{du}{u} = 0,$$

it follows that

$$\int_\Gamma \mathrm{div}(u', v')dt = \int_\Gamma uf'(u)\phi(v)dt.$$

Referring to figure 2.3, we shall divide the circuit Γ:

$$\int_\Gamma uf'(u)\phi(v)dt = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} + \int_{\gamma_5} + \int_{\gamma_6}\right) uf'(u)\phi(v)dt,$$

where the paths γ_i, $i = 1, \dots, 6$ represent respectively the arcs from A to P, from P to Q passing through R, from Q to B, from Q to B', from Q' to P' passing through L, from P' to A.

With respect to Cheng's notation, if we replace λ with σ, S with u, x with v, $\frac{K-a}{2}$ with m, and we consider any concave parabola vanishing in b_1 and b_2, whose vertex belongs to the line $u = m$, we can notice that all estimates in his proof keep valid for our integral, except those dealing with paths γ_1, γ_3, γ_4 and γ_6.

In particular, let us consider:

$$\left(\int_{\gamma_1} + \int_{\gamma_6}\right) uf'(u)\phi(v)dt = \left(\int_{\gamma_1} + \int_{\gamma_6}\right) \frac{f'(u)\phi(v)}{f(u)\phi(v) - v}du =$$

$$= \int_{u_A}^{u_P} \frac{f'(u)\phi(v_2(u))}{f(u)\phi(v_2(u)) - v_2(u)}du +$$

$$+ \int_{u_A}^{u_P} \frac{f'(2m - \sigma - u)\phi(v_1(2m - \sigma - u))}{f(2m - \sigma - u)\phi(v_1(2m - \sigma - u)) - v_1(2m - \sigma - u)}du,$$

where $v_1(u)$ and $v_2(u)$ are the restrictions of the periodic orbit $v(u)$ to arcs γ_1 and γ_6 respectively. Since $f'(u) = -f'(2m - \sigma - u)$ and $f(u) = f(2m - \sigma - u)$ for symmetry, the last integral turns out to be equal to:

$$\int_{u_A}^{u_P} \frac{(f'(u))[v_2(u)\phi(v_1(2m - \sigma - u)) - v_1(2m - \sigma - u)\phi_2(v_2(u))]}{(f(u)\phi(v_2(u)) - v_2(u))(f(u)\phi(v_1(2m - \sigma - u)) - v_1(2m - \sigma - u))}du;$$

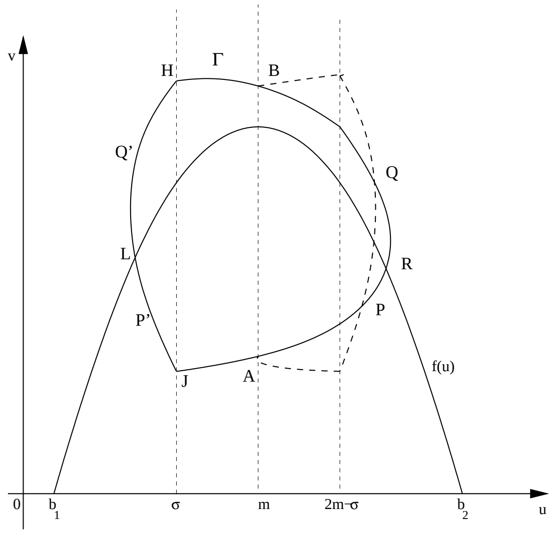

Figure 1: The limit cycle Γ and the notation for the proof of Theorem 2.4

$f'(u)$ is negative if $u \in [u_A, u_P]$, whereas the denominator is certainly positive if the two products $f(\cdot)\phi(\cdot)$ are bigger than $f(u)$, so the condition $\phi(v) \geq 1$ for all $v \geq 0$ is sufficient. Finally, if ϕ is a nonincreasing function, since $v_2(u) > v_1(2m - \sigma - u)$ (see [2]), the numerator is positive too, and the integral can be evaluated as negative. An analogous idea can be applied to the remaining integrals on the arcs γ_3 and γ_4. Consequently, if $\phi(v)$ is nonincreasing and bigger than one, $\int_\Gamma \operatorname{div}(u', v')dt < 0$, and this directly implies the stability of the closed orbit Γ. But since two adjacent periodic orbits cannot be positively stable on the sides facing each other (e.g. [3] p. 397), the limit cycle of system (1.2) must be unique. □

§3. Conclusions and further developments

The dynamics associated to a Rosenzweig-Mac Arthur modified ODE modelling a demand governed market for innovative commodities may be particularly rich and give rise to interesting cyclical phenomena. As it often happens in qualitative analysis of economic models, specific properties of the involved functions may yield results clarifying the behaviour of the interactive agents. As far as this dynamic system is concerned, we believe that the hypotheses we formulated constitute a realistic pattern.

However, it would be interesting to deepen this analysis, for instance by weakening the hypotheses on the growth rate function, in order to investigate which qualitative properties may be preserved when making use of more general dynamics.

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