Existence and regularity of optimal solution for a dead oil isotherm problem

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Abstract. We study a system of nonlinear partial differential equations resulting from the traditional modelling of oil engineering within the framework of the mechanics of a continuous medium. Existence and regularity of the optimal solutions for this system is established.

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§1. Introduction

We are interested to the existence and regularity of optimal solution for the following "dead oil isotherm" problem:

$$
\begin{cases}
\partial_t u - \Delta\varphi(u) = div\,(g(u)\nabla p) & \text{in } Q_T = \Omega \times (0,T)\,, \\
\partial_t p - div\,(d(u)\nabla p) = f & \text{in } Q_T = \Omega \times (0,T)\,, \\
u|_{\partial\Omega} = 0\,, \quad u|_{t=0} = u_0\,, & \\
p|_{\partial\Omega} = 0\,, \quad p|_{t=0} = p_0\,, &
\end{cases}
\tag{1.1}
$$

where Ω is an open bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary.

Equations (1.1) serve as a model of an incompressible biphasic flow in a porous medium, with applications in the industry of exploitation of hydrocarbons. To understand the optimal control problem we consider here, some words about the recovery of hydrocarbons are in order. At the time of the first run of a layer, the flow of the crude oil towards the surface is due to the energy stored in the gases under pressure or in the natural hydraulic system. To mitigate the consecutive decline of production and the decomposition of the site, water injections are carried out, before the normal exhaustion of the layer. The water is injected through wells with high pressure, by pumps specially drilled with this end. The pumps allow the displacement of the crude oil towards the wells of production. The wells must be judiciously distributed, which gives rise to a difficult problem of optimal control: how to choose the best installation sites of the production wells? The cost functional to be minimized comprises all the important parameters that intervene in the processes.

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Existence and uniqueness to the system (1.1), for the case when the term $\partial_t p$ is missing but for more general boundary conditions, is established in [3]. Optimal control of systems governed by partial differential equations is investigated in literature by many authors, we can refer to [2, 7, 9]. To study existence and regularity of solutions which provide Gâteaux differentiability of the nonlinear operator corresponding to (1.1), we are forced to assume more regularity on the control f as well as to impose compatibility conditions between initial and boundary conditions. The considered cost functional comprises four terms and has the form

$$J(u,p,f) = \frac{1}{2}\|u-U\|^2_{2,Q_T} + \frac{1}{2}\|p-P\|^2_{2,Q_T} + \frac{\beta_1}{2}\|f\|^{2q_0}_{2q_0,Q_T} + \frac{\beta_2}{2}\|\partial_t f\|^2_{2,Q_T} \tag{1.2}$$

where $1 < q_0 < 2$, $\beta_1 > 0$ and $\beta_2 > 0$ are two coefficients of penalization; U and P are given data. Here u is the reduced saturation of the phase oil at the moment t. The initial saturation is known and p is the total pressure. The first two terms in (1.2) make possible to minimize the difference between the reduced saturation of oil and a given U, respectively the global pressure and a known initial pressure P. We remark that the choice of the objective functional is not unique. We can always add further terms of penalization to take into account other properties which one may want to control. The paper is organized as follows. In Section we set up the notation, the functional spaces and some important lemmas used throughout the work. Section is devoted to the existence of optimal solutions. We obtain necessary estimates on the sequence minimizing the cost functional which allows us to pass to the limit. Finally, in Section we establish a regularity theorem.

§2. Notation and functional spaces

In the sequel we suppose that φ, g and d are real valued C^1-functions satisfying:

(H1) $0 < c_1 \le d(r),\ \varphi(r) \le c_2;\ |d'(r)|,\ |\varphi'(r)|,\ |\varphi''(r)| \le c_3 \quad \forall r \in \mathbb{R}$.

(H2) $u_0,\ p_0 \in C^2\left(\bar{\Omega}\right)$, $U,\ P \in L^2(Q_T)$, where $u_0,\ p_0 : \Omega \to \mathbb{R}$, $U,\ P : Q_T \to \mathbb{R}$, and $u_0|_{\partial\Omega} = p_0|_{\partial\Omega} = 0$.

We consider the following spaces:

$$W^{1,0}_p(Q_T) := L^p\left(0,T,W^1_p(\Omega)\right) = \{u \in L^p(Q_T),\ \nabla u \in L^p(Q_T)\}\ ,$$

endowed with the norm $\|u\|_{W^{1,0}_p(Q_T)} = \|u\|_{p,Q_T} + \|\nabla u\|_{p,Q_T}$;

$$W^{2,1}_p(Q_T) := \left\{u \in W^{1,0}_p(Q_T),\ \nabla^2 u\,, \partial_t u \in L^p(Q_T)\right\}\ ,$$

with the norm $\|u\|_{W^{2,1}_p(Q_T)} = \|u\|_{W^{1,0}_p(Q_T)} + \left\|\nabla^2 u\right\|_{p,Q_T} + \|\partial_t u\|_{p,Q_T}$;

$$V := \left\{u \in W^{1,0}_2(Q_T),\ \partial_t u \in L^2\left(0,T,W^{-1}_2(\Omega)\right)\right\}\ .$$

We now state some important lemmas that are used later. Lemma 2.1 is needed in the proof of our existence result.

Lemma 2.1 ([1]). *Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain with a C^1-boundary, and a matrix $A(x,t) = (A_{ij}(x,t))$ satisfying the conditions*

$$\exists \gamma_0 > 0 \text{ such that } A_{ij}(x,t)\xi_i\xi_j \geq \gamma_0|\xi|^2 \quad \forall \xi \in \mathbb{R}^n\,, \\ A_{ij} \in L^\infty(Q_T)\,, \quad A_{ij} = A_{ji}\,. \tag{2.3}$$

Assume also that $f \in L^{2q_0}(Q_T)$, $u_0 \in W^1_{2q_0}(\Omega)$ for some $q_0 > 1$ and let $u \in C\left([0,T];\, L^2(\Omega)\right) \cap W_2^{1,0}(Q_T)$ be a weak solution to the equation

$$\partial_t u - div\,(A(x,t)\nabla u) = f \text{ in } Q_T\,, \\ u|_{\partial\Omega} = 0\,, \quad u|_{t=0} = u_0\,. \tag{2.4}$$

Then, there exists a constant $q > 1$, depending on n, q_0, γ_0, Q_T, and $\|A\|_{\infty,Q_T}$, such that $u \in W^{1,0}_{2q}(Q_T)$ and the estimate

$$\|\nabla u\|_{2q,Q_T} \leq c\left(\|f\|_{2q,Q_T} + \|u_0\|_{W^1_{2q}(\Omega)}\right)$$

holds.

We use the following two lemmas to get some regularity of weak solutions.

Lemma 2.2 (De Giorgi-Nash-Ladyzhenskaya-Uraltseva theorem [6]). *Assume $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$ a C^1-bounded domain; let $f \in L^{s,r}(Q_T) = L^s(0,T,L^r(\Omega))$, $u_0 \in C^\alpha(\bar\Omega)$ for some $\alpha_0 > 0$, $u_0|_{\partial\Omega} = 0$ and*

$$\frac{1}{r} + \frac{n}{2s} < 1\,.$$

Assume (2.3) holds and let $u \in W_2^{1,0}(Q_T)$ be a weak solution of (2.4). Then, there exists $\alpha > 0$ such that $u \in C^{\alpha,\frac{\alpha}{2}}(\bar Q_T)$ and

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar Q_T)} \leq c\left(\|f\|_{L^{s,r}(Q_T)} + \|u_0\|_{C^\alpha(\bar\Omega)}\right)\,.$$

Lemma 2.3 ([6]). *For any function $u \in C^{\alpha,\frac{\alpha}{2}}(\bar Q_T) \cap L^2\left(0,T;\, \overset{\circ}{W}{}^1_2(\Omega) \cap W^2_2(\Omega)\right)$ there exist numbers N_0, ϱ_0 such that for any $\varrho \leq \varrho_0$ there is a finite covering of Ω by sets of the type $\Omega_\varrho(x_i)$, $x_i \in \bar\Omega$, such that the total number of intersections of different $\Omega_{2\varrho}(x_i) = \Omega \cap B_{2\varrho}(x_i)$ does not increase N_0. Hence, we have the estimate*

$$\|\nabla u\|^4_{4,Q_T} \leq c\,\|u\|^2_{C^{\alpha,\frac{\alpha}{2}}(\bar Q_T)}\,\varrho^{2\alpha}\left(\left\|\nabla^2 u\right\|^2_{2,Q_T} + \frac{1}{\varrho^2}\,\|\nabla u\|^2_{2,Q_T}\right)\,.$$

§3. Existence of the optimal solution

We denote by (P) the problem of minimizing (1.2) subject to (1.1) in the class $(u,p,f) \in W_1^{2,1}(Q_T) \times V \times L^2(Q_T)$.

Theorem 3.4. *Under hypotheses (H1)-(H2) there is a $q > 1$, depending on the data of the problem, such that there exists an optimal solution $\left(\bar{u}, \bar{p}, \bar{f}\right)$ of problem (P) verifying:*

$$\bar{u} \in W_q^{2,1}(Q_T)\,,$$
$$\bar{p} \in C\left([0,T]; L^2(\Omega)\right) \cap W_{2q}^{1,0}(Q_T)\,, \quad \partial_t \bar{p} \in L^2\left(0, T, W_2^{-1}(\Omega)\right)\,,$$
$$\bar{f} \in L^{2q_0}(Q_T)\,, \quad \partial_t \bar{f} \in L^2(Q_T)\,.$$

Proof. Let $(u^m, p^m, f^m) \in W_1^{2,1}(Q_T) \times V \times L^{2q_0}(Q_T)$ be a sequence minimizing $J(u,p,f)$. Then we have

$$(f^m) \text{ is bounded in } L^{2q_0}(Q_T),$$
$$(\partial_t f^m) \text{ is bounded in } L^2(Q_T).$$

Using the parabolic equation governed by the global pressure p and Lemma 2.1, we know that there exists a number $q > 1$ such that

$$\|\nabla p^m\|_{2q,Q_T} \leq \left(\|f^m\|_{2q,Q_T} + \|u_0\|_{W_{2q}^1(\Omega)}\right).$$

Multiplying the second equation of (1.1) by p, using the hypotheses and Young's inequality, we get

$$\sup_t \|p^m\|_{2,\Omega}^2 + \|\nabla p^m\|_{2q,Q_T}^2 \leq c\|f^m\|_{2,Q_T}^2.$$

Furthermore, we have that $\partial_t p^m$ is bounded in $L^2(0,T;W_2^{-1}(\Omega))$. By Aubin's Lemma [8], (p^m) is compact in $L^2(Q_T)$. Using now the first equation of (1.1) we have

$$\partial_t u^m - \varphi'(u^m)\triangle u^m - \varphi''(u^m)|\nabla u^m|^2 = div(g(u^m)\nabla p^m).$$

Hence

$$\|u^m\|_{W_q^{2,1}(Q_T)} \leq c,$$

where all the constants c are independent of m. Using the Lebesgue theorem and the compacity arguments of J. L. Lions [8] we can extract subsequences, still denoted by $(p^m), (u^m)$ and (f^m), such that

$$p^m \to \overline{p} \text{ weakly in } W_{2q}^{1,0}(Q_T),$$
$$\partial_t p^m \to \partial_t \overline{p} \text{ weakly in } L^2(0,T;W_2^{-1}(\Omega),$$
$$p^m \to \overline{p} \text{ strongly in } L^2(Q_T),$$
$$p^m \to \overline{p} \text{ a.e. in } L^2(Q_T),$$
$$u^m \to \overline{u} \text{ a.e. in } L^2(Q_T),$$
$$f^m \to \overline{f} \text{ weakly in } L^{2q_0}(Q_T),$$
$$\partial_t f^m \to \partial_t \overline{f} \text{ weakly in } L^2(Q_T).$$

The existence of an optimal solution $(\overline{u}, \overline{p}, \overline{f})$ follows, in a standard way, by passing to the limit in problem (1.1) and by using the fact that J is lower semicontinuous with respect to the weak convergence. □

§3. The regularity of solutions

We now prove some regularity to the solutions predicted by Theorem 3.4.

Theorem 4.5. *Suppose that (H1) and (H2) are satisfied and let* $\left(\bar{u}, \bar{p}, \bar{f}\right)$ *be an optimal solution of our problem* (P)*. Then, there exist a* $\alpha > 0$ *such that the following regularity conditions are verified:*

$$\bar{p} \in C^{\alpha, \frac{\alpha}{2}}\left(\bar{Q}_T\right), \tag{4.5}$$

$$\bar{u},\, \bar{p} \in W_4^{1,0}(Q_T), \tag{4.6}$$

$$\bar{u},\, \bar{p} \in W_2^{2,1}(Q_T), \tag{4.7}$$

$$\partial_t \bar{u},\, \partial_t \bar{p} \in L^\infty\left(0, T; L^2(\Omega)\right) \cap W_2^{1,0}(Q_T), \tag{4.8}$$

$$\bar{u} \in C^{\frac{1}{4}}(Q_T), \tag{4.9}$$

$$\bar{u} \in W_{2q_0}^{2,1}(Q_T), \quad \bar{p} \in W_{2q_0}^{2,1}(Q_T). \tag{4.10}$$

Proof. First, we remark that (4.5) is an immediate consequence of Lemma 2.2. To show the other results, we begin by proving the following lemma.

Lemma 4.6. *Consider* (u, p, f) *solution of (1.1). Assume that hypotheses (H1) and (H2) hold. Then,*

$$\sup_{t \in (0,T)} \|\nabla p\|_{2,\Omega}^2 + \left\|\nabla^2 p\right\|_{2,Q_T}^2 \le c\left(\|\nabla p\|_{4,Q_T}^4 + \|\nabla u\|_{4,Q_T}^4\right) + c$$

where c *depend on* u_0 *and* f*.*

Proof. ¿From the second equation of (1.1) we have

$$\partial_t p - d(u)\Delta p = d'(u)\nabla u.\nabla p + f\,.$$

Multiplying this equation by $\partial_t p$ and integrating over Ω, we obtain

$$\|\partial_t p\|_2^2 + \frac{c}{2}\frac{\partial}{\partial t}\|\nabla p\|^2 \le c\int_\Omega |\nabla p \nabla u \partial_t p|\, dx + \int_\Omega |f \partial_t p|\, dx\,.$$

Using Young's inequality and integrating in time, we get the desired estimate. □

To continue the proof of Theorem 4.5 we need to estimate $\|\nabla u\|_{4,Q_T}$ in function of $\|\nabla p\|_{4,Q_T}$. Then, taking into account the first equation of (1.1), it is well known that $u \in W_4^{1,\frac{1}{2}}(Q_T)$ and

$$\|\nabla u\|_{4,Q_T} \le c\|\nabla p\|_{4,Q_T} \tag{4.11}$$

(see [4]). Using Lemma 2.3, we have that for any $\varrho < \varrho_0$

$$\|\nabla p\|_{4,Q_T}^4 \le c\,\|p\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}^2\,\varrho^{2\alpha}\left\{\|\nabla p\|_{4,Q_T}^4 + \frac{1}{\varrho^2}\|\nabla p\|_{2,Q_T}^2\right\} + C_{u_0,p_0,f_0}\,.$$

Calling Lemma 2.2, we then get (4.6) for an eligible choice of ϱ. After using (4.11) we obtain that $u \in W_4^{1,0}(Q_T)$. On the other hand, we have by the first equation of

(1.1) and (4.6) that $u \in W_2^{2,1}(Q_T)$. Moreover, it follows by Lemma 4.6 and the fact that $u \in W_2^{2,1}(Q_T)$ that $p \in W_2^{2,1}(Q_T)$.

Now, in order to prove (4.8), we differentiate both equations of (1.1) with respect to time:

$$\begin{aligned} \partial_{tt}u - div\,(\varphi'(u)\nabla\partial_t u) - div\,(\varphi''(u)\nabla\partial_t u\nabla u) \\ = div\,(g'(u)\partial_t u\nabla p) + div\,(g(u)\nabla\partial_t p)\,, \end{aligned} \tag{4.12}$$

$$\partial_{tt}p - div\,(d(u)\nabla\partial_t p) - div\,(d'(u)\nabla\partial_t u\nabla p) = \partial_t f. \tag{4.13}$$

Multiplying (4.13) by $\partial_t p$ and integrating over Ω we get

$$\frac{\partial}{\partial t}\|\partial_t p\|_{2,\Omega}^2 + c\|\partial_t\nabla p\|_{2,\Omega}^2 \le c_f + c\|\partial_t p\|_{2,\Omega}^2 + c\int_\Omega |\partial_t u\nabla p\nabla\partial_t p|\,dx.$$

By Young's inequality we have

$$\begin{aligned} \int_\Omega |\partial_t u\nabla p\nabla\partial_t p|\,dx &\le \|\partial_t u\nabla p\|_{2,\Omega}\|\partial_t\nabla p\|_{2,\Omega} \\ &\le c\|\partial_t u\nabla p\|_{2,\Omega}^2 + \frac{c}{2}\|\partial_t\nabla p\|_{2,\Omega}^2. \end{aligned}$$

On the other hand, by Holder's inequality we obtain

$$\begin{aligned} \|\partial_t u\nabla p\|_{2,\Omega}^2 &= \int_\Omega |\partial_t u|^2|\nabla p|^2 \\ &\le \left(\int_\Omega |\partial_t u|^4\right)^{\frac{1}{2}}\left(\int_\Omega |\partial_t p|^4\right)^{\frac{1}{2}} = \|\partial_t u\|_{4,\Omega}^2\|\nabla p\|_{4,\Omega}^2. \end{aligned}$$

Using the following multiplicative inequality [5]

$$\|\partial_t u\|_{4,\Omega}^2 \le c\|\partial_t u\|_{2,\Omega}\|\partial_t\nabla u\|_{2,\Omega} \quad \forall u \in W_2^1(\Omega),$$

we obtain

$$\begin{aligned} \int_\Omega |\partial_t u\nabla p\nabla\partial_t p|\,dx &\le c\|\partial_t u\|_{4,\Omega}^2\|\nabla p\|_{4,\Omega}^2 + \frac{c}{2}\|\partial_t\nabla p\|_{2,\Omega}^2 \\ &\le c\|\partial_t u\|_{2,\Omega}\|\partial_t\nabla u\|_{2,\Omega}\|\nabla p\|_{4,\Omega}^2 + \frac{c}{2}\|\partial_t\nabla p\|_{2,\Omega}^2 \\ &\le c\|\partial_t u\|_{2,\Omega}^2\|\nabla p\|_{4,\Omega}^4 + c\|\partial_t\nabla u\|_{2,\Omega}^2 + \frac{c}{2}\|\partial_t\nabla p\|_{2,\Omega}^2. \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial}{\partial t}\|\partial_t p\|_{2,\Omega}^2 + c\|\partial_t\nabla p\|_{2,\Omega}^2 \\ \le c_f + c\|\partial_t p\|_{2,\Omega}^2 + c\|\partial_t\nabla u\|_{2,\Omega}^2 + c\|\partial_t u\|_{2,\Omega}^2\|\nabla p\|_{4,\Omega}^4. \end{aligned} \tag{4.14}$$

Multiplying (4.12) by $\partial_t u$ and integrating over Ω, we get

$$\frac{\partial}{\partial t}\|\partial_t u\|_{2,\Omega}^2 + c\|\partial_t \nabla u\|_{2,\Omega}^2 \\ \leq -\int_\Omega \varphi''(u)\partial_t u \nabla u \partial_t \nabla u - \int_\Omega g'(u)\partial_t u \nabla p \nabla u - \int_\Omega g(u)\partial_t \nabla p \nabla u.$$

Similar as before, we have:

$$\left|\int_\Omega \varphi''(u)\partial_t u \nabla u \partial_t \nabla u\right| \leq c\|\partial_t u\|_{2,\Omega}^2 \|\nabla u\|_{4,\Omega}^4 + c\|\partial_t \nabla u\|_{2,\Omega}^2,$$

$$\left|\int_\Omega g'(u)\partial_t u \nabla p \nabla u\right| \leq c\|\partial_t u\|_{2,\Omega}^2 \|\nabla p\|_{4,\Omega}^4 + c\|\nabla u\|_{2,\Omega}^2,$$

$$\left|\int_\Omega g(u)\partial_t \nabla p \nabla u\right| \leq c\|\partial_t \nabla p\|_{2,\Omega}^2 + c\|\nabla u\|_{2,\Omega}^2.$$

It follows by using (4.11) that

$$\frac{\partial}{\partial t}\|\partial_t u\|_{2,\Omega}^2 + c\|\partial_t \nabla u\|_{2,\Omega}^2 \leq c\|\partial_t u\|_{2,\Omega}^2 \|\nabla p\|_{4,\Omega}^4 + c\|\partial_t \nabla p\|_{2,\Omega}^2 + c\|\nabla u\|_{2,\Omega}^2. \tag{4.15}$$

Calling (4.14) and (4.15) together, it yields

$$\frac{d}{dt}\left\{\|\partial_t u\|_{2,\Omega}^2 + \|\partial_t p\|_{2,\Omega}^2\right\} + \|\partial t \nabla u\|_{2,\Omega}^2 + \|\partial t \nabla p\|_{2,\Omega}^2 \\ \leq c\left(1 + \|\nabla p\|_{4,\Omega}^4\right)\left\{\|\partial_t u\|_{2,\Omega}^2 + \|\partial_t p\|_{2,\Omega}^2\right\} + c_{u_0,p_0,f}.$$

We thus obtain (4.8) by applying Gronwall lemma. On the other hand, we have

$$\|\partial_t u\|_{4,\Omega}^2 \leq c\|\partial_t u\|_{2,\Omega}\|\partial_t \nabla u\|_{2,\Omega},$$

and from (4.8) we obtain $\partial_t u \in L^4(Q_T)$. Using (4.6) and the fact that $W_4^1(Q_T) \hookrightarrow C^{\frac{1}{4}}(\overline{Q_T})$, the regularity estimate (4.9) follows. Finally, the right hand side of the first equation of (1.1) belongs to $L^4(Q_T) \hookrightarrow L^{2q_0}(Q_T)$ as $2q_0 \leq 4$. Using thus (4.7) we get $u \in W_{2q_0}^{2,1}(Q_T)$. Since $f \in L^{2q_0}(Q_T)$, the same estimate follows for p from the second equation of the system (1.1) and we conclude with (4.10). □

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