# Proper homothetic vector fields in Bianchi Type-I Space-Time

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**Abstract.** A study of proper homothetic vector field in Bianchi type-1 space-times is given by using direct integration technique. Using the above mentioned technique we show that the above space-times admit proper homothetic vector field in a very special choice of $f$, $k$ and $h$. The dimensions of homothetic algebra are 4, 5, 7 and 11.

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## 1 Introduction

Throughout $M$ is representing the four dimensional, connected, hausdorff space-time manifold with Lorentz metric $g$ of signature $(-,+,+,+)$. The curvature tensor associated with $g_{ab}$, through Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$, and the Ricci tensor components are $R_{ab} = R^c{}_{acb}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol $L$, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively.

Any vector field $X$ on $M$ can be decomposed as

$$X_{a;b} = \frac{1}{2}h_{ab} + F_{ab} \tag{1.1}$$

where $h_{ab} = h_{ba}$ and $F_{ab} = -F_{ba}$ are symmetric and skew-symmetric tensor on $M$, respectively. If

$$h_{ab} = \alpha g_{ab}, \quad \alpha \in R$$

equivalent

$$g_{ab,c}X^c + g_{bc}X^c_{,a} + g_{ac}X^c_{,b} = \alpha g_{ab} \tag{1.2}$$

then $X$ is called a homothetic vector field on $M$. If $X$ is homothetic and $\alpha \neq 0$ then it is called proper homothetic while $\alpha = 0$ it is Killing [3, 4]. Further consequences and geometrical interpretations of (1.2) are explored in [2, 5]. It also follows from (1.2) that [2, 4]

$$L_X R^a{}_{bcd} = 0 \qquad L_X R_{ab} = 0.$$

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## 2 Main results

A Bianchi type-1 space-time is a spatially homogeneous space-time which admits an abelian Lie algebra of isometries $G_3$, acting on spacelike hypersurfaces, generated by the spacelike Killing vector fields which are:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}. \tag{2.1}$$

The line element in usual coordinate system is [7]

$$ds^2 = -dt^2 + f(t)dx^2 + k(t)dy^2 + h(t)dz^2, \tag{2.2}$$

where $f$, $k$ and $h$ are some nowhere zero functions of $t$ only. The possible Segre type of the above space-time is $\{1, 111\}$ or one of its degeneracies. A vector field $X$ is said to be a homothetic vector field if it satisfy equation (1.2). One can write (1.2) explicitly using (2.2) we have

$$X^0_{,0} = \frac{\alpha}{2}, \tag{2.3}$$

$$fX^1_{,0} - X^0_{,1} = 0, \tag{2.4}$$

$$kX^2_{,0} - X^0_{,2} = 0, \tag{2.5}$$

$$hX^3_{,0} - X^0_{,3} = 0, \tag{2.6}$$

$$\dot{f}\, X^0 + 2fX^1_{,1} = \alpha f, \tag{2.7}$$

$$kX^2_{,1} + fX^1_{,2} = 0, \tag{2.8}$$

$$hX^3_{,1} + fX^1_{,3} = 0, \tag{2.9}$$

$$\dot{k}\, X^0 + 2kX^2_{,2} = \alpha k, \tag{2.10}$$

$$hX^3_{,2} + kX^2_{,3} = 0, \tag{2.11}$$

$$\dot{h}\, X^0 + 2hX^3_{,3} = \alpha h. \tag{2.12}$$

Equations (2.3), (2.4), (2.5) and (2.6) give

$$\left.\begin{aligned} X^0 &= \tfrac{\alpha}{2}t + A^1(x, y, z) \\ X^1 &= A^1_x(x, y, z) \int \tfrac{1}{f} dt + A^2(x, y, z) \\ X^2 &= A^1_y(x, y, z) \int \tfrac{1}{k} dt + A^3(x, y, z) \\ X^3 &= A^1_z(x, y, z) \int \tfrac{1}{h} dt + A^4(x, y, z) \end{aligned}\right\}, \tag{2.13}$$

where $A^1(x, y, z)$, $A^2(x, y, z)$, $A^3(x, y, z)$ and $A^4(x, y, z)$ are functions of integration. In order to determine $A^1(x, y, z)$, $A^2(x, y, z)$, $A^3(x, y, z)$ and $A^4(x, y, z)$ we need to integrate the remaining six equations. To avoid lengthy calculations here we will only present results for full details see [1].

**Case (1)** Four independent homothetic vector fields:

In this case the space-time (2.2) takes the form

$$ds^2 = -dt^2 + (\alpha t + 2d_{10})^{2(1-\frac{2}{\alpha}d_5)}dx^2 + dy^2 + (\alpha t + 2d_{10})^{2(1-\frac{2}{\alpha}d_{14})}dz^2, \tag{2.14}$$

and homothetic vector fields in this case are

$$\left.\begin{array}{l} X^0 = t\frac{\alpha}{2} + d_{10} \\ X^1 = xd_5 + d_6 \\ X^2 = y\frac{\alpha}{2} + d_{12} \\ X^3 = zd_{14} + d_{15} \end{array}\right\}, \tag{2.15}$$

where $d_5, d_6, d_{10}, d_{12}, d_{14}, d_{15} \in R$. The above space-time (2.14) admits four independent homothetic vector fields in which three are Killing vector fields which are given in (2.1) and one is proper homothetic vector field which is

$$Z^1 = (t, 0, y, 0). \tag{2.16}$$

**Case (2)** Four independent homothetic vector fields:

In this case the space-time (2.2) takes the form

$$ds^2 = -dt^2+(\alpha t+2d_{10})^{2(1-\frac{2}{\alpha}d_5)}dx^2+(\alpha t+2d_{10})^{2(1-\frac{2}{\alpha}d_{11})}dy^2+(\alpha t+2d_{10})^{2(1-\frac{2}{\alpha}d_{14})}dz^2. \tag{2.17}$$

Homothetic vector fields in this case are

$$\left.\begin{array}{l} X^0 = t\frac{\alpha}{2} + d_{10} \\ X^1 = xd_5 + d_6 \\ X^2 = yd_{11} + d_{12} \\ X^3 = zd_{14} + d_{15} \end{array}\right\}, \tag{2.18}$$

where $d_5, d_6, d_{10}, d_{11}, d_{12}, d_{14}, d_{15} \in R$. The above space-time (2.17) admits four independent homothetic vector fields in which three are Killing vector fields which are given in (2.1) and one is proper homothetic vector field which is

$$Z^2 = (t, 0, 0, 0). \tag{2.19}$$

**Case (3)** Five independent homothetic vector fields:

In this case the space-time (2.2) takes the form

$$ds^2 = -dt^2 + (\alpha t + 2d_4)^{2(1-\frac{2}{\alpha}d_7)}dx^2 + (dy^2 + dz^2), \tag{2.20}$$

and homothetic vector fields in this case are

$$\left.\begin{array}{l} X^0 = t\frac{\alpha}{2} + d_4 \\ X^1 = xd_5 + d_8 \\ X^2 = y\frac{\alpha}{2} + zd_5 + d_6 \\ X^3 = z\frac{\alpha}{2} - yd_5 + d_3 \end{array}\right\}, \tag{2.21}$$

where $d_3, d_4, d_5, d_6, d_7, d_8 \in R$. The above space-time (2.20) admits five independent homothetic vector fields in which four are Killing vector fields and one is proper homothetic vector field which is

$$Z^3 = (t, 0, y, z). \tag{2.22}$$

**Case (4)** Five independent homothetic vector fields:

In this case the space-time (2.2) takes the form

$$ds^2 = -dt^2 + (\alpha t + 2d_4)^{2(1-\frac{2}{\alpha}d_7)}dx^2 + (\alpha t + 2d_4)^{2(1-\frac{2}{\alpha}d_{12})}(dy^2 + dz^2). \tag{2.23}$$

Homothetic vector fields in this case are

$$\left.\begin{aligned} X^0 &= t\tfrac{\alpha}{2} + d_4 \\ X^1 &= xd_7 + d_8 \\ X^2 &= y\tfrac{\alpha}{2} + zd_5 + d_{11} \\ X^3 &= zd_{12} - yd_5 + d_{13} \end{aligned}\right\}, \tag{2.24}$$

where $d_4, d_5, d_7, d_8, d_{11}, d_{12}, d_{13} \in R$. The above space-time (2.24) admits five independent homothetic vector fields in which four are Killing vector fields and one is proper homothetic vector field which is

$$Z^4 = (t, 0, y, 0). \tag{2.25}$$

**Case (5)** Seven independent homothetic vector fields:

In this case the space-time (2.2) takes the form

$$ds^2 = -dt^2 + (\alpha t + 2d_9)^{2(1-\frac{2}{\alpha}d_8)}(dx^2dy^2 + dz^2). \tag{2.26}$$

It follows from [6] homothetic vector fields in this case are

$$\left.\begin{aligned} X^0 &= t\tfrac{\alpha}{2} + d_9 \\ X^1 &= xd_8 - yd_{20} - zd_{18} + d_{21} \\ X^2 &= yd_8 + xd_{20} + zd_{17} + d_{23} \\ X^3 &= zd_8 + xd_{18} - yd_{17} + d_{24} \end{aligned}\right\}, \tag{2.27}$$

where $d_8, d_9, d_{17}, d_{18}, d_{20}, d_{21}, d_{23}, d_{24} \in R$. The above space-time (2.26) admits seven independent homothetic vector fields in which six are Killing vector fields and one is proper homothetic vector field which is

$$Z^5 = (t, 0, 0, 0). \tag{2.28}$$

**Case (6)** Eleven independent homothetic vector fields:

In this case the above space-time (2.2) becomes Minkowski space-time

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2.29}$$

homothetic vector fields in this case are

$$\left.\begin{aligned} X^0 &= t\tfrac{\alpha}{2} + xd_{11} + yd_{10} + zd_{12} + d_{13} \\ X^1 &= x\tfrac{\alpha}{2} + td_{11} + yd_{17} + zd_{15} + d_{18} \\ X^2 &= y\tfrac{\alpha}{2} + td_{10} + zd_{14} - xd_{17} + d_{19} \\ X^3 &= z\tfrac{\alpha}{2} + td_{12} - xd_{15} - yd_{14} + d_{16} \end{aligned}\right\}, \tag{2.30}$$

where $d_{10}, d_{11}, d_{12}, d_{13}, d_{14}, d_{15}, d_{16}, d_{17}, d_{18}, d_{19} \in R$. The above space-time (2.29) admits eleven independent homothetic vector fields in which ten are Killing vector fields and one is proper homothetic vector field which is

$$Z^6 = (t, x, y, z). \tag{2.31}$$

## Summary

In this paper a study of Bianchi type-I space-times according to their proper homothetic vector fields is given by using the direct integration technique. From the above study we obtain the following:
**(i)** The space-times which admit four independent homothetic vector fields are given in equations (2.14) and (2.17) (see for details cases (1) and (2)).
**(ii)** The space-times which admit five independent homothetic vector fields are given in equations (2.20) and (2.23) (see for details cases (3) and (4)).
**(iii)** The space-time which admits seven independent homothetic vector fields are given in equation (2.26) (see for details case (5)).
**(iv)** The space-time which admits eleven independent homothetic vector fields are given in equation (2.29) (see for details case (6)).


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