A fixed point theorem on the fuzzy number space

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Abstract. In this article, a fixed point theorem for a contractive and a nonlinear contractive map on fuzzy number space is proved.

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1 Preliminaries

The definition of fuzzy number is given by many mathematicians such as Zadeh [9], Kramosil and Michalek [6], Kaleva and Seikala [4] and [5], etc. In addition, the existence of a fixed point in a fuzzy metric space has been studied in recent years and also there is a broad set of applications (see [1], [7], [8], etc). It is obvious that most of the fixed point theorems are hold in the complete metric spaces (i.e. in the metric spaces which every Cauchy sequence is convergent). In this article, we prove a fixed point theorem in a space of fuzzy numbers which is not complete. It means that the completeness of the space is omitted and the existence and uniqueness of a fixed point are proved. In order to do this, we recall some concepts and results which will be required in the sequel.

Definition 1.1. A fuzzy number is a fuzzy set $u : \mathbb{R} \to I = [0, 1]$ which satisfies
(i) u is upper semi-continuous.
(ii) $u(x) = 0$ outside some interval $[c, d]$.
(iii) There are real numbers a, b such that $c \leq a \leq b \leq d$ and,
(iii)-(1) $u(x)$ is monotonic increasing on $[c, a]$.
(iii)-(2) $u(x)$ is monotonic decreasing on $[b, d]$.
(iii)-(3) $u(x) = 1,\ a \leq x \leq b$.
The set of all these fuzzy numbers is denoted by E.

Definition 1.2. For $1 \leq p < \infty$,

$$d_p(u, v) = (\int_0^1 d_H([u]^\alpha, [v]^\alpha)^p d\alpha)^{\frac{1}{p}},$$

for all $u, v \in E^n$, where $[.]^\alpha$ is for α-cut of fuzzy numbers.

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Definition 1.3. $d_H(A, B)$ is the Hausdorff distance between nonempty subsets A and B of $\mathbb{R}^n$ by

$$d_H(A, B) = \max\{d_H^\star(A, B), d_H^\star(B, A)\},$$

where

$$d_H^\star(A, B) = \sup\{d(b, A) : b \in B\},$$

and

$$d(x, A) = \inf\{||x - a|| : a \in A\}.$$

Note that (E^n, d_p) is a metric space for $1 \leq p < \infty$, but it is not a complete metric space (see [2] for more details).

In the next section, we prove an iterated theorem. This theorem proves the existence of a fixed point for a contractive map and a nonlinear contractive map.

2 Fixed point under contractive map

In this section, the definition of a contractive map and a nonlinear contractive map are rewritten and an iterative theorem is proved. This theorem shows that the existence of a convergent subsequence of an iterate sequence (of a contractive map) proves the existence of a fixed point. In order to do this, we recall the definition of a contractive map on E^n as follows:

Definition 2.1. Let E^n be as before and $D = d_p$ for some $1 \leq p < \infty$. The mapping $f : E^n \to E^n$ is called contractive (or nonexpansive), if

$$D(f(u), f(v)) < D(u, v), \tag{2.1}$$

for all $u \neq v \in E^n$.

Now we have our main theorem as follows:

Theorem 2.2. *Let (E^n, D) be as above, and f a contractive mapping of E^n into itself such that there exists a fuzzy number $u \in E^n$ whose sequence of iterates $(f^n(u))$ contains a convergent subsequence $(f^{n_i}(u))$; then $\xi = \lim_{i\to\infty} f^{n_i}(u) \in E^n$ is a unique fixed point of f.*

Proof. Suppose $f(\xi) \neq \xi$ and consider the sequence $(f^{n_i+1}(u))$ which, it can easily be verified, converges to $f(\xi)$.

The mapping $r(u, v)$ of $Y = E^n \times E^n$ into the real line defined by

$$r(u, v) = \frac{D(f(u), f(v))}{D(u, v)}, \tag{2.2}$$

for all $u \neq v$. Note that f is a contractive map of E^n into itself, and also D is continuous on $E^n \times E^n$. Thus r is a continuous function on Y. This shows that there exists a neighborhood U of $(\xi, f(\xi)) \in Y$ such that $u, v \in U$ implies

$$0 \leq r(u, v) < R < 1. \tag{2.3}$$

Let $B_1 = B_1(\xi, \rho)$ and $B_2 = B_2(f(\xi), \rho)$ be open neighborhoods centered at ξ and $f(\xi)$, respectively, and of radius $\rho > 0$ small enough so as to have

$$\rho \leq \frac{1}{3} D(\xi, f(\xi)), \tag{2.4}$$

and $B_1 \times B_2 \subset U$.
By the assumption, there exists a positive integer N such that $i > N$ implies $f^{n_i}(u) \in B_1$ and hence by (2.1) also $f^{n_i+1}(u) \in B_2$.
Thus, by (2.4)

$$D(f^{n_i}(u), f^{n_i+1}(u)) > \rho, \ (i > N) \tag{2.5}$$

On the other hand, for such i, it follows from (2.2) and (2.3) that

$$D(f^{n_i+1}(u), f^{n_i+2}(u)) < RD(f^{n_i}(u), f^{n_i+1}(u)). \tag{2.6}$$

A repeated use of (2.6) for $l > j > N$ now gives

$$\begin{aligned} D(f^{n_l}(u), f^{n_l+1}(u)) &\leq D(f^{n_{l-1}+1}(u), f^{n_{l-1}+2}(u)) \\ &< RD(f^{n_{l-1}}(u), f^{n_{l-1}+1}(u)) \\ &\leq \cdots \\ &< R^{l-j} D(f^{n_j}(u), f^{n_j+1}(u)) \to 0, \quad \text{as} \quad l \to \infty. \end{aligned}$$

Which is contradiction with the property (2.5). Thus $f(\xi) = \xi$ and this means that ξ is a fixed point of f.

In order to prove the uniqueness of ξ, suppose there is a $\eta \neq \xi$ with $f(\eta) = \eta$, then it follows that

$$D(\xi, \eta) = D(f(\xi), f(\eta)) < D(\xi, \eta),$$

which is contradiction. This proves the uniqueness and, thus, accomplishes the proof of this theorem. □

Now assume that the mapping $\psi : [0, \infty) \to [0, \infty))$ is upper semi-continuous from the right, $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

Definition 2.3. Let (E^n, D) be as in Definition 2.1 and ψ be as above. We call the mapping $f : E \to E$ is nonlinear contractive, if

$$D(f(u), f(v)) \leq \psi(D(u, v)),$$

for all $u \neq v \in E$.

Theorem 2.4. *Let (E^n, D) be as in Definition 2.1, and f a nonlinear contractive mapping of E^n into E^n, such that there exists a fuzzy number $u \in E^n$ whose sequence of iterates $(f^n(u))$ contains a convergent subsequence $(f^{n_i}(u))$, then $\xi = \lim_{i \to \infty} f^{n_i}(u) \in E^n$ is a unique fixed point of f.*

Proof. With respect to the property of ψ, that is, $\psi(t) < t$ for all $t > 0$, we obtain

$$D(f(u), f(v)) < \psi(D(u, v)) < D(u, v).$$

Thus by Theorem 2.2, the proof is complete. □

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