Mechanical Lagrangian systems with external forces

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Abstract. We consider mechanical Lagrangian systems with external forces that depend also on velocity. The geometrical model is given by a special semispray S. We show that when the system is dissipative the energy of the system decreases on the integral curves of S. We give also sufficient condition that the system be stable using a Lyapunov function constructed with the energy of the system.

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1 Mechanical Lagrangian systems

A mechanical system has the total energy as sum of kinetic energy and potential energy. The kinetic energy $\frac{1}{2}mv^2$ has in the rectangular coordinates $(x^1, ..., x^n)$ the expression

$$\frac{1}{2}m[(\dot{x}^1)^2 + (\dot{x}^2)^2 + ... + (\dot{x}^n)^2]$$

where $\dot{x}^i = \frac{dx^i}{dt}$.

If we pass to other coordinates called in the analytic mechanics generalized coordinates (q^i), the form of the kinetic energy becomes $\frac{1}{2}mg_{ij}(q)\dot{q}^i\dot{q}^j$, a positively defined quadratic form with the coefficients $g_{ij}(q)$.

If the configuration space of the system is a differentiable manifold M, the kinetic energy appears as a Riemannian metric on M. And the potential energy is a function V on the manifold M.

These facts justify:

Definition 1.1. *A Lagrangian mechanical system is a triplet (M, g, V), with g a Riemannian metric on M and V a real function on M.*

Let $\tau : TM \to M$ be the tangent bundle over M.

The function $L : TM \to \mathbf{R}$,

$$L(V_q) = \frac{1}{2}g(V_q, V_q) - (V \circ \tau)(V_q)$$

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is called the *Lagrangian* of the system (M, g, V). We identify $V \circ \tau$ with V.

A differential curve $\gamma : I \to M,\ t \to \gamma(t)$, with I an open interval in $\mathbf{R}$, is called trajectory of the Lagrangian system (M, g, V) if the function $\gamma(t)$ is a solution of Euler - Lagrange equation:

1.1 $$\frac{d}{dt}\left(\frac{\partial L(\dot\gamma(t))}{\partial \dot\gamma}\right) - \frac{\partial L}{\partial \gamma} = 0.$$

Written in the coordinates (x^i, y^i) on TM with the equation of the curve $t \to x^i(t)$, this equation becomes:

1.1′ $$\frac{d}{dt}\left(\frac{\partial L(x^i, \dot x^i)}{\partial \dot x^i}\right) - \frac{\partial L}{\partial x^i} = 0,\ \dot x^i = \frac{dx^i}{dt}.$$

Inserting L as above and computing we obtain:

$$g_{ij}\ddot x^j + \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)\dot x^j \dot x^k = -\frac{\partial V}{\partial x^i}.$$

Multiplying by (g^{hi}) one yields:

1.2 $$\ddot x^h + \Gamma^h_{jk}(x)\dot x^j \dot x^k = -g^{hi}\frac{\partial V}{\partial x^i}$$

where Γ^h_{jk} are Christoffel coefficients derived from (g_{ij}).

If we denote by ∇ the Levi-Civita connection of the manifold (M, g) the equations (1.2) can be written in the form

1.3 $$\nabla_{\dot\gamma}\dot\gamma = F \circ \gamma,\ F = -\text{grad}V.$$

The situation described above is the simplest and for this reason, the triplet (M, g, V) is also called simple mechanical system. The vector F is also called external force.

If $F = 0$, the equation of the trajectory is $\nabla_{\dot\gamma}\dot\gamma = 0$ and coincides with the equations of the geodesics of g.

On the other hand, $\nabla_{\dot\gamma}\dot\gamma$ is the acceleration vector and (1.3) represents Newton's equation $m\overrightarrow{a} = \overrightarrow{F}$, (with $m = 1$).

Frequently, the external force has also a non-gradient component that is $F = \text{grad}V + R$ and the equation (1.3) has the form:

1.4 $$\nabla_{\dot\gamma}\dot\gamma = -(\text{grad}V) \circ \gamma + R \circ \gamma$$

We notice that when describing the dynamic of the simple mechanical system given by the Euler - Lagrange equations, the most important role is of the Lagrangian function $L(x, y)$. Thus is natural the following change of Definition 1.1:

Definition 1.2. *It is called a Lagrangian system a triplet (M, L, F), where L is a Lagrangian and F is a vector field.*

For L a regular Lagrangian, we will use the metric $g_{ij}(x, y) = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i \partial y^j}$ and its inverse (g^{jk}) in order to pass from F to a covector field and conversely.

2 Stability and dissipation for Lagrangian mechanical systems with external forces

We continue the study of the Lagrangian mechanical systems with problems concerning dissipation and stability.

Let $\sum = (M, L(x,y), F(x,y))$ be a Lagrangian system with M a differentiable manifold, L a regular Lagrangian and $F(x,y) = (F_i(x,y))$ external forces seen as a $d-$covector field on TM.

We will postulate that the evolution equations of the system $\sum$ are the following:

2.1 $$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = F_i(x,y);\ y^i = \frac{dx^i}{dt}.$$

Expanding the derivative with respect to ..., replacing the derivatives $\dfrac{\partial^2 L}{\partial y^i \partial y^j}$ with $2g_{ij}$ and multiplying with (g^{jk}) the equation (2.1) becomes:

2.2 $$\frac{d^2x^i}{dt^2} + 2G^i(x,\dot{x}) = \frac{1}{2}F^i(x,\dot{x}),$$

where

2.2′ $$g^i(x,y) = \frac{1}{4}g^{ij}\left(\frac{\partial^2 L}{\partial y^j \partial x^k}y^k - \frac{\partial L}{\partial x^i}\right),\ y = \dot{x} = \frac{dx^i}{dt}.$$

With the notation $y^i = \dfrac{dx^i}{dt}$, equations (2.2) are equivalent with the system of equations:

2.3 $$\begin{aligned} &\frac{dx^i}{dt} = y^i \\ &\frac{dy^i}{dt} = -2\left(G^i - \frac{1}{4}F^i\right) \end{aligned}.$$

The solutions of this system can be seen as integrable curves of the vector field S^* on TM given by

2.4 $$S^* = y^i\frac{\partial}{\partial x^i} - 2G^{*i}(x,y)\frac{\partial}{\partial y^i},\ G^{*i} = G^i - \frac{1}{4}F^i.$$

This is a semispray.For the theory of semisprays we refer to [1].

The semispray associated to Lagrangian L is

2.4′ $$S = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}.$$

Definition 2.1. *The mechanical system $\sum$ is dissipative if the force F is dissipative i.e. $F_j(x,y)y^j \leq 0$ and $\sum$ is strictly dissipative if the force F is strictly dissipative i.e. $F_j(x,y)y^j \leq -\alpha y_j y^j$, with $\alpha > 0$ and $y_j = g_{ij}(x,y)y^i$.*

The conditions of dissipation and of strictly dissipation can be also formulated as follows: if the matrix $(g_{ij}(x,y))$ is positively defined, it defines a Riemannian metric g in the vertical bundle over TM. The force field (F^i) can be seen as a section in vertical bundle by the definition $F = F^i(x,y)\dfrac{\partial}{\partial y^i}$. The Liouville field $C = y^i\dfrac{\partial}{\partial y^i}$ appears as a section in vertical bundle and we have $g(C,C) = g_{ij}y^iy^j = y_iy^j := \|y\|^2$. Also, we have $F_j(x,y)y^j = g(F,C)$ and so, the dissipation condition one writes $g(F,C) \le 0$ and the strictly dissipation condition becomes $g(F,C) \le -\alpha\|y\|^2$.

Theorem 2.1. *If the Lagrangian system $\sum$ is dissipative then its energy $E(x,y) = y^i\dfrac{\partial L}{\partial y^i} - L$ decreases on the solutions curves of the equations (2.3). If the system $\sum$ is strictly dissipative and the solutions curves have not singularities, then the energy E is strictly decreasing on solutions curves of the equations (2.3).*

Proof. Let $\gamma : t \to (x(t), y(t))$, $y = \dot{x}$ a curve on TM solution of the system (2.3). Along this curve, we have

$$\frac{dE}{dt} = \ddot{x}\frac{\partial L}{\partial \dot{x}^i} + \dot{x}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \ddot{x}\frac{\partial L}{\partial \dot{x}^i} - \dot{x}^i\frac{\partial L}{\partial x^i} -$$

$$-\ddot{x}\frac{\partial L}{\partial \dot{x}^i} = \dot{x}^i\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}\right) =$$

$$= \dot{x}^iF_i(x,\dot{x}) \le 0.$$

So, E is decreasing on γ.

If $\sum$ is strictly dissipative, then $\dfrac{dE}{dt} = \dot{x}^iF_i(x,\dot{x}) \le -\alpha\|\dot{x}\|^2 < 0$ so E is strictly decreasing on γ.□

We will define the equilibrium point of the system $\sum$ as zeros of the semispray S^*.

From (2.4) it results that the equilibrium points of the system $\sum$ are $(x_0^i, 0)$ or $O_{x_0} \in T_{x_0}M$, where (x_0^i) must be a solution of the equations

$$G^{*i}(x_0^i, 0) = 0 \Leftrightarrow G^i(x_0^i, 0) - \frac{1}{4}F^i(x_0^i, 0) = 0.$$

For a Lagrange manifold (M, L), the tangent manifold TM is a Riemannian manifold assuming that $g_{ij}(x,y) = \dfrac{\partial^2 L}{\partial y^i \partial y^j}$ is positively defined.

The Riemannian metric on TM is

$$g_L = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$$

with

$$\delta y^i = dy^i + N_j^i dx^i$$

where (N_j^i) are the coefficients of the nonlinear connection defined by the semispray: $N_j^i = \dfrac{\partial G^i}{\partial y^j}$.

If the manifold (TM, G) is complete as a metric space one can be given theorems of stability for the equilibrium points of a vector field on TM, similar to those from $\mathbf{R}^n$.

In [2] one introduces the notion of Lyapunov functions for X in a zero x_0 as a real function V on M with the properties:

1) $V(x_0) = 0$, $V(x) > 0$, $\forall x \neq x_0$

2) $\mathbf{L}_x V(x) \leq 0 \Leftrightarrow X(V(x)) \leq 0$, $\forall x \in M$

and one proves

Proposition 2.1.([2])*If there exists a Lyapunov function for X in x_0, then x_0 is a stable point.*

We will apply this proposition for the manifold TM and the vector field S^*. We get

Theorem 2.2. *Let $\sum = (M, L, F)$ be a dissipative Lagrangian system with (TM, g_L) a complete Riemannian manifold. Let $(x_0^i, 0)$ be an equilibrium point of $\sum$, that is a zero of the vector field S^*. We suppose that $(x_0^i, 0)$ is a point of absolute minimum for the energy E of the system $\sum$. Then $(x_0^i, 0)$ is a stable equilibrium point.*

Proof. We consider the function $\widetilde{E}(x, y) = E(x, y) - E(x_0, 0)$.

We notice that $E(x_0, 0) = -L(x_0, 0)$ so that $\widetilde{E}(x, y) = E(x, y) + L(x, 0)$.

We have

i) $\widetilde{E}(x_0, 0) = 0$, $\widetilde{E}(x, y) > 0$ because $E(x_0, 0)$ is the absolute minimum value for E.

ii) $\mathbf{L}_{S^*}(\widetilde{E}) = \mathbf{L}_{S^*}(E) = y^i \dfrac{\partial E}{\partial x^i} - 2G^* \dfrac{\partial E}{\partial y^i} = y^i \dfrac{\partial E}{\partial x^i} - 2G^i \dfrac{\partial E}{\partial y^i} + \dfrac{1}{2} F^i \dfrac{\partial E}{\partial y^i}$.

But $\dfrac{\partial E}{\partial y^i} = \dfrac{\partial L}{\partial y^i} + y^j \dfrac{\partial^2 L}{\partial y^j \partial y^i} - \dfrac{\partial L}{\partial y^i} = 2 g_{ij} y^j$.

Using this, we obtain:

$$\mathbf{L}_{S^*}(\widetilde{E}) = y^i \left(\frac{\partial^2 L}{\partial x^i \partial y^j} y^j - \frac{\partial L}{\partial x^i} \right) - 4 g_{ij} y^j G^j + F^i g_{ij} y^j.$$

The first two terms from the right side of this equality cancel each other and we get:

$$\mathbf{L}_{S^*}(\widetilde{E}) = F^i g_{ij} y^j = F^j g_{ij} y^i = F_i y^i \leq 0$$

from the dissipation condition on $\sum$.

So, $\widetilde{E}$ is a Lyapunov function for S^* in $(x_0, 0)$.

By the Proposition 2.1, $(x_0, 0)$ is a stable equilibrium point.□

Theorem 2.3. *Let $\sum = (M, L, F)$ be a dissipative Lagrangian system with (TM, g_L) a complete Riemannian manifold, $L \geq 0$ and $L(x, y)$ a homogeneous function of degree $m \geq 2$ in the variables y. Let $(x_0^i, 0)$ a point on TM, with $F^i(x_0^i, 0) = 0$. Then, $(x_0^i, 0)$ is a stable equilibrium point of S^*.*

Proof. Directly we observe that the functions $G^i(x, y)$ are homogeneous of degree 2 in y variables. It results that $G^i(x_0^i, 0) = 0$ and from the hypothesis $F^i(x_0^i, 0) = 0$, we infer that $(x_0^i, 0)$ is an equilibrium point for S^*. From Euler's theorem we have $y^i \dfrac{\partial L}{\partial y^i} = mL$, and the energy $E = mL - L = (m-1)L$. So it results:

i) $E(x_0^i, 0) = 0$, $E \geq 0$.

ii) $\mathbf{L}_{S^*}(E) = y^i \dfrac{\partial E}{\partial y^i} - 2G^i \dfrac{\partial E}{\partial y^i} + \dfrac{1}{2} F^i \dfrac{\partial E}{\partial y^i} = F_i y^i \leq 0$.

The equalities are obtained in the same way as in the proof of Theorem 2.1 and the inequality takes place because of the hypothesis that $\sum$ is a dissipative system.

Thus, E is Lyapunov function for S^* and the equilibrium point $(x_0^i, 0)$. By the Proposition 2.1, $(x_0^i, 0)$ is a stable equilibrium point.□

References

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