Upper top spaces

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Abstract. In this paper a method for constructing new top spaces by using of universal covering spaces of special Lie subsemigroups of a top space is presented. As a result a generalization of the notion of fundamental group which is a completely simple semigroup is deduced. The persistence of MF-semigroups under isomorphisms of top spaces is proved.

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1 Introduction

Top spaces are generalization of Lie groups [3]. In this paper we begin with a top space and then by use of it we will construct an upper top space for it.
Let us to recall the definition of a top space [3, 4].
A top space is a smooth manifold T (not necessary connected) admitting an operation called multiplication, subject to the set of rules given below:
(i) $(xy)z = x(yz)$ for all x, y, z in T (associative law).
(ii) For each x in T there exists a unique z in T such that $xz = zx = x$, we denote z by $e(x)$ (existence and uniqueness of identity).
(iii) For each x in T there exists y in T such that $xy = yx = e(x)$ (existence of inverse).
(iv) The mapping $m_1 : T \to T$ is defined by $m_1(u) = u^{-1}$ and the mapping $m_2 : T \times T \to T$ is defined by $m_2(u_1, u_2) = u_1u_2$ are smooth maps.
(v) $e(xy) = e(x)e(y)$ for all $x, y \in T$.
The properties (i), (ii), and (iii) imply that T is a completely simple semigroup [1].
If T is a top space then $T = \bigcup_{t\in T} T_{e(t)}$ where $T_{e(t)} = \{s \in T : e(s) = e(t)\}$. Moreover for each $t \in T$, $T_{e(t)}$ with the differentiable structure and product of T is a Lie group.
Let T and S be two top spaces and let $f : T \to S$ be an algebraic homomorphism, i.e., $f(xy) = f(x)f(y)$ for all $x, y \in T$. Then $f(e(x)) = e(f(x))$ and $f : T_{e(x)} \to S_{e(f(x))}$ is a group homomorphism, where $x \in T$. The kernel of f defined by $Kerf = \bigcup_{t\in T} kerf_t$ where f_t is the restriction of f on $T_{e(t)}$ [4]. We will use of this notion to present a generalization of the notion of fundamental group as the kernel of covering map of an

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upper top space of a given top space. We will show, the persistence of this structure under the isomorphisms of top spaces.

2 A method for constructing new top spaces

In this section we assume that for all $t \in T$, the set $T_{e(t)}$ is a connected set [5]. If $(\tilde{T}_{e(t)}, p_t, \tilde{e(t)})$ is a universal covering space of $(T_{e(t)}, e(t))$, then $\tilde{T}_{e(t)}$ is a Lie group with the multiplication $\tilde{m_t}(\tilde{t_1}, \tilde{t_2})$ with $\tilde{t_1}, \tilde{t_2} \in \tilde{T}_{e(t)}$ such that $p_t o \tilde{m}_t(\tilde{t_1}, \tilde{t_2}) = m_t(p_t(\tilde{t_1}, \tilde{t_2})$ where m_t is the restriction of m on $T_{e(t)} \times T_{e(t)}$.
Let $\tilde{T}$ be the disjoint union of $\tilde{T}_{e(t)}$ where $t \in T$. Then we define the product $\tilde{m}$ on $\tilde{T} \times \tilde{T}$ such that $p_{st} o \tilde{m}(\tilde{s}, \tilde{t}) = m(p_s(\tilde{s}), p_t(\tilde{t}))$ and $\tilde{m}(\tilde{e(s)}, \tilde{e(t)}) = \tilde{e(st)}$.

Theorem 2.1 $\tilde{m}$ is determined uniquely by the above equalities.

Proof. If $\tilde{s}, \tilde{t} \in \tilde{T}$ then $m(p_s(\tilde{s}), p_t(\tilde{t}))$ is a unique member of $T_{e(st)}$. Since p_{st} is a local diffeomorphism on the connected component $T_{e(st)}$ then $\tilde{m}(\tilde{s}, \tilde{t})$ determines uniquely. q.e.d.

Theorem 2.2 $(\tilde{T}, \tilde{m})$ is a top space.

Proof. If $\tilde{r}, \tilde{s}, \tilde{t} \in \tilde{T}$ then
$p_{r(st)} o(\tilde{m} o(id_{\tilde{T}} \times \tilde{m})) = mo(p_r \times p_{st}) o(id_{\tilde{T}} \times \tilde{m}) = mo(p_r \times (p_{st} o \tilde{m})) = mo(p_r \times mo(p_s \times p_t)) = mo(id_T \times m)o(p_r \times p_s \times p_t)$.
Moreover we have
$p_{r(st)} o(\tilde{m} o(\tilde{m} \times id_{\tilde{T}})) = p_{(rs)t} o(\tilde{m} o(\tilde{m} \times id_{\tilde{T}})) = mo(m \times id_T)o(p_r \times p_s \times p_t)$.
So
$p_{r(st)} o(\tilde{m} o(id_{\tilde{T}} \times \tilde{m})) = p_{r(st)} o(\tilde{m} o(\tilde{m} \times id_{\tilde{T}}))$.
Moreover
$\tilde{m} o(id_{\tilde{T}} \times \tilde{m})(\tilde{e(r)}, \tilde{e(s)}, \tilde{e(t)}) = \tilde{m} o(\tilde{m} \times id_{\tilde{T}})(\tilde{e(r)}, \tilde{e(s)}, \tilde{e(t)})$.
Thus
$\tilde{m} o(id_{\tilde{T}} \times \tilde{m}) = \tilde{m} o(\tilde{m} \times id_{\tilde{T}})$.
For $u \in T_{e(v)}$ we have
$p_v(\tilde{m}(\tilde{u}, \tilde{e(v)})) = m(p_v(\tilde{u}, p_{e(v)}(\tilde{e(v)})) = m(p_v(\tilde{u}), e(v)) = p_v(\tilde{u})$
and
$\tilde{m}(\tilde{e(v)}, \tilde{e(v)}) = \tilde{e(v)}$.
So $\tilde{m}(\tilde{v}, \tilde{e(v)}) = \tilde{v}$ for all $\tilde{v} \in \tilde{T}$.
Now for given $t \in T$ let $\tilde{i} : \tilde{T} \to \tilde{T}$ be the lifting of the mapping $iop_t : \tilde{T} \to T$ with $\tilde{i}(\tilde{e(t)}) = \tilde{e(t)}$ where i is the inverse map of T (i.e. $i(t) = t^{-1}$).
Since
$p_t o(\tilde{m}(\tilde{t}, \tilde{i}(\tilde{t})) = mo(p_t \times p_t)(\tilde{t}, \tilde{i}(\tilde{t})) = m(p_t(\tilde{t}), po\tilde{i}(\tilde{t})) = m(p_t(\tilde{t}), (p_t(\tilde{t}))^{-1}) = e(t)$
and $\tilde{m}(\tilde{e(t)}, \tilde{i}(\tilde{e(t)})) = \tilde{e(t)}$, then $\tilde{m}(\tilde{t}, \tilde{i}(\tilde{t})) = \tilde{e(t)}$.
The set $\tau_{\tilde{T}} = \{\tilde{U} \subseteq \tilde{T} : \tilde{U} \cap \tilde{T_{e(t)}}$ is open in $\tilde{T_{e(t)}}$ for all $t \in T\}$ is a topology for $\tilde{T}$. With this topology we can extend the differentiable structure of $\tilde{T_{e(t)}}$ on $\tilde{T}$ and with this differentiable structure $\tilde{T}$ is a top space. q.e.d.
The straightforward calculations show that the mapping $p : \tilde{T} \to T$ defined by $p(\tilde{t}) = p_t(\tilde{t})$ is a homomorphism of top spaces.
The pair $(\tilde{T}, p)$ is called the upper top space of T.

3 A generalization of fundamental groups

We begin this section with the following theorem.

Theorem 3.1 If $(\tilde{T}, p)$ and $(\tilde{S}, q)$ be two upper top spaces of a top space T, then $Kerp$ is isomorphic to $Kerq$.

Proof. We know that $\tilde{T}$, and $\tilde{S}$ are disjoint unions of $\tilde{T_{e(t)}}$ and $\tilde{S_{e(t)}}$ respectively, where $(\tilde{T}_{e(t)}, p_t, \tilde{e(t)})$ and $(\tilde{S}_{e(t)}, q_t, \tilde{e(t)})$ are universal covering spaces of $(T_{e(t)}, e(t))$ for all $t \in T$. So for all $t \in T$ there exists a diffeomorphism $g_t : \tilde{T}_{e(t)} \to \tilde{S}_{e(t)}$ such that $q_t o g_t = p_t$. Now we define $g : Kerp \to Kerq$ by $g(\tilde{x}) = g_x(\tilde{x})$, we show that g is an isomorphism. If $\tilde{x} \in Kerp$ then $p_x(\tilde{x}) = e(x)$. Thus $q_x o g_x(\tilde{x}) = e(x)$. Hence $g_x(\tilde{x}) \in Kerq$. If $\tilde{y} \in Kerq$ then $q_y(\tilde{y}) = e(y)$. So $p_y(g_y^{-1}(\tilde{y}) = q_y o g_y(g_y^{-1}(\tilde{y}) = e(y)$. Thus $g_y^{-1}(\tilde{y} \in \tilde{T}_y \cap Kerp$. Since $Kerp$ and $Kerq$ are disjoint unions of $kerp_t$ and $kerq_t$ respectively, and $g_t : kerp_t \to kerq_t$ is an isomorphism, then g is an isomorphism. q.e.d.

We now define the main notion of this section.

Definition 3.1 If $(\tilde{T}, p)$ is an upper top space of T then the $Kerp$ is called the MF-semigroup of T.

The next theorem shows that MF-semigroups are generalization of fundamental groups.

Theorem 3.2 If T is a top space and D is the MF-semigroup of it then D is isomorphic to $\overset{o}{\bigcup}_{t\in e(T)} \pi_1(T_{e(t)}, e(t))$ where $\pi_1(T_{e(t)}, e(t))$ is the fundamental group of $T_{e(t)}$ with the base point $e(t)$, and $\overset{o}{\bigcup}$ denotes the disjoint union.

Proof. The definition of D implies that $D = Kerp$, where $(\tilde{T}, p)$ is an upper top space of T. So $D = \bigcup_{t\in e(T)} kerp_t$. Since for all $t \in T$, $T_{e(t)}$ is a Lie group then $kerp_t \cong \pi_1(T_{e(t)}, e(t))$ [2]. Thus $D \cong \overset{o}{\bigcup}_{t\in T} \pi_1(T_{e(t)}, e(t))$. q.e.d.

Definition 3.2 If T and U are two top spaces, then a mapping $f : T \to U$ is called an isomorphism if it is an algebraic isomorphism and a C^∞ diffeomorphism. Two top spaces are called isomorphic if there is an isomorphism between them.

Theorem 3.3 Let D and E be MF-semigroups of top spaces T and U respectively. Moreover let T and U be isomorphic top spaces. Then D and E are isomorphic semigroups.

Proof. Suppose $f : T \to U$ be an isomorphism, then $f(e(t)) = e(f(t))$ for all $t \in T$. Thus $f(T_{e(t)}) = U_{e(f(t))}$. Since f is a diffeomorphism then $\pi_1(T_{e(t)}, e(t)) \cong \pi_1(U_{e(f(t))}, e(f(t)))$. Thus theorem 3.2 implies that $D \cong E$. q.e.d.

Conclusion. If for all $t \in T$, $T_{e(t)}$ is connected and open set then the notion of an upper top space of T is close to the notion of universal space. More precisely if T is a Lie group then the upper top space of T is the universal covering of $(T, 1)$. Moreover theorem 3.2 implies that $Kerp \cong \pi_1(T, 1)$.

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