On finite circular spaces

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Abstract. In this paper, we give some combinatorial properties of finite circular spaces which are circle regular, and two characterizations of inversive planes by using circular spaces.

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1 Introduction

In this section, we give some basic definitions and concepts used in this paper.

Definition. A set $\mathcal{P}$ whose elements are called points and a set $\mathcal{L}$ of certain subsets of $\mathcal{P}$ whose elements are called lines and $\circ \subseteq \mathcal{P} \times \mathcal{L}$. The incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \circ)$ is called a linear space if:

L1. Every line contains at least two points.

L2. Any two points belong to a unique line.

If $|\mathcal{P}|$ and $|\mathcal{L}|$ are finite then $\mathcal{S}$ is called finite. If every line has k points and every point is on r lines then, the linear space is called (k, r)-regular.

It is known that line regularity of a linear space implies point regularity [1]. Finite linear spaces have been studied in detaily by many mathematicians and it has been obtained very nice results [1]-[8].

In this paper, we define the concept of a finite circular space similar to the concept of a finite linear space. Firstly, we prove some propositions which establish connections between linear spaces and circular spaces and then we want to characterize inversive planes by using circular spaces.

Definition. Let $\mathcal{P}$ be a set of points, $\mathcal{C}$ be a set of certain distinguished subsets of points called circles and $\circ \subseteq \mathcal{P} \times \mathcal{C}$. The incidence structure $\mathbf{C} = (\mathcal{P}, \mathcal{C}, \circ)$ is called a circular space if:

C1. Every circle contains at least three distinct points.

C2. Any three distinct points are contained in exactly one circle.

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If $|\mathcal{P}|$ and $|\mathcal{C}|$ are finite then $\mathbf{C}$ is called finite. A circular space $\mathbf{C}$ is said to be circle regular if every circle has the same number of points and $\mathbf{C}$ is said to be point regular if every point is on the same number of circles.

Definition. An inversive plane (or a Möbius plane) $\mathbf{I}$ is a collection of points and distinguished subsets of points called circles satisfying the following axioms:

I1. Any three distinct points are contained in exactly one circle.

I2. If c is a circle such that $q \circ c$, $p \phi c$ for two points p,q, then there is exactly one circle $c^{'}$ pass through p and tangent to c at q. (Two circles are called tangent if they have exactly one point in common).

I3. There are at least two circles and every circle contains at least three distinct points.

It is clear that every inversive plane is a circular space.

We shall concern with some important types of circles set of $\mathbf{I}$ (or the circular space $\mathbf{C}$). The set of all circles pass through two distinct points p, q is called a boundle $[p, q]$ and these points are called carriers of the boundle $[p, q]$. The set of all circles which have only point r in common is called as a pencil. Where this point is called carrier of the pencil. Alternatively, any point p and any circle c with $p \circ c$ or any pair of tangent circles c, c' also determine a pencil uniquely and are denoted by $\langle p, c\rangle$ or $\langle c, c'\rangle$ respectively. A flock is a set $\mathcal{F}$ of mutually disjoint circles in $\mathbf{I}$ (or $\mathbf{C}$) such that with the exception of precisely two points p, q, every point of $\mathbf{I}$ is on a (necessarily unique) circle of $\mathcal{F}$. These points are again called the carriers of flock. (Dembowski [5]) .

2 Some connections between linear spaces and circular spaces

Now, we give a clear connection between linear spaces and circular spaces.

Proposition 2.1. *If $\mathbf{C} = (\mathcal{P}, \mathcal{C}, \circ)$ is a circular space, $p \in \mathcal{P}$ and $\mathcal{L} = \{\ l \subset \mathcal{P} : l \cup \{p\} \in \mathcal{C}\}$, then $\mathbf{C}_p = (\mathcal{P} \backslash \{\ p\}, \mathcal{L}, \circ)$ is a linear space.*

Proof. *Let l be any line of $\mathcal{L}$. $|\ l \cup \{p\}| \geq 3$, since $l \cup \{p\} \in \mathcal{C}$ by C1. So $|\ l| \geq 2$, that is, L1 holds in $\mathbf{C}_p$. Let q and r be any two distinct points. q and r are on just one line in $\mathbf{C}_p$, since p, q and r are on just one circle in $\mathbf{C}$,that is L2 holds. So $\mathbf{C}_p$ is a linear space.* □

Proposition 2.2. *Let $\mathbf{C} = (\mathcal{P}, \mathcal{C}, \circ)$ be a finite circular space, $p \in \mathcal{P}$, $c \in \mathbf{C}$, $p \phi c$. Then $r_p \geq \binom{k_c}{2}$ where r_p is the total number of circles on p and k_c is the total number of points on c.*

Proof. *Let q and r be any points on c distinct from p. Since p, q, r are on just one circle of $\mathbf{C}$ by C2, $r_p \geq \binom{k_c}{2}$.*□ □

Proposition 2.3. *Let $\mathbf{C} = (\mathcal{P}, \mathcal{C}, \circ)$ be a circular space whose every circle contains exactly $n+1$ points and $|\ \mathcal{C}\ |= b > 1$. Let c be a circle and q, r be any two points with $r \circ c \phi q$ and k be total number of circles through q tangent to c at r.*

(i) *If* $k = 0$ *and* $n \geq 3$*, then* $\mathbf{C}_p$ *is a projective plane of order* $n-1$ *where* $n \in \{3, 5, 11\}$.

(ii) *If* $k = 1$ *and* $n \geq 2$*, then* $\mathbf{C}_p$ *is an affine plane of order* n *(in this case,* $\mathbf{C}$ *is an inversive plane).*

(iii) *If* $k, n \geq 2$ *and* $n \geq 1 + \sqrt{1+k}$*, then* $\mathbf{C}_p$ *is a hyperbolic plane.*

Proof. *(i) Let* $k = 0$ *and* $n \geq 3$*.Then any two circles on* p *intersect in a point different from* p*, that is, any two lines of* $\mathbf{C}_p$ *intersect.*

Let s, t *be two distinct points of* $\mathbf{C}_p$*.* st *is a line in* $\mathbf{C}_p$*, since* s, t, p *belong to unique circle in* $\mathbf{C}$ *by C2. There exists a set of four points no three of which are collinear by* $n \geq 3$*. Thus* $\mathbf{C}_p$ *is a projective plane of order* $n-1$*, since there are* n *points on every line in* $\mathbf{C}_p$*. On the otherhand, one can write*

$$b = \left[(n-1)^2 + (n-1) + 2\right]\left[(n-1)^2 + (n-1) + 1\right]/(n+1).$$

So it must be $(n+1)|12$*, since* $b \in \mathbb{N}$*, that is, the order of projective plane* $\mathbf{C}_p$ *is* $n-1 = 2$*,* $n-1 = 4$ *or* $n-1 = 10$*.*

(ii) If $k = 1$*,* $n \geq 2$*, then it is well known that* $\mathbf{C}_p$ *is an affine plane of order* n*, since* $\mathbf{C}$ *is an inversive plane of order* n *in this case (Dembowski [6]).*

(iii) Let maximum and minimum number of lines through a point in $\mathbf{C}_p$ *be* r_M *and* r_m*, and, maximum and minimum number of points on a line in* $\mathbf{C}_p$ *be* k_M *and* k_m *with* $k, n \geq 2$*,* $n \geq 1 + \sqrt{1+k}$*.*

If $r_m \geq k_M + 2$ *and* $k_m(k_m - 1) \geq r_M$*, then it is well known that* $\mathbf{C}_p$ *is a hyperbolic plane in the sense of Graves [7](Bumcrot [3]).*

It is clear that $k_m = k_M = n$ *and* $r_m = r_M = n+k$ *by the definition of* $\mathbf{C}_p$*. Hence* $r_M \geq k_M + 2$*.* $n \geq 1 + \sqrt{1+k}$ *implies* $k_m(k_m - 1) \geq n + k$*.* □

Remark 2.1. *If* $n-1$ *is an odd integer in the Proposition 2.3 (*i*), all circles of* $\mathbf{C}$ *is concurrent at the point* p*. In fact that* $\mathbf{C}_p$ *is a projective plane, if there exists at least one circle* c' *not through* p*, then* c' *is a hyperoval in* $\mathbf{C}_p$*. But a projective plane of odd order does not contain any hyperoval, that is,* c' *must be on* p*. Thus* $\mathbf{C}$ *contains only one circle(this is degenerate case!).*

Proposition 2.4. *Let* $\mathbf{C} = (\mathcal{P}, \mathcal{C}, \circ)$ *be a circular space in which every circle contains* $n+1$ *points with* $n \in \{3, 5, 11\}$*,* $k = 0$*. Then,*

(i) *There are exactly* $(n-1)^2 + (n-1) + 2$ *points in* $\mathbf{C}$*.*

(ii) *There are exactly* $(n-1)^2 + (n-1) + 1$ *circles on each point of* $\mathbf{C}$*.*

(iii) *The total number of circles is*

$$\left[(n-1)^2 + (n-1) + 2\right]\left[(n-1)^2 + (n-1) + 1\right]/(n+1).$$

(iv) *Each boundle contains* n *circles.*

(v) $\mathcal{C}$ *does not contain tangent circles.*

(vi) Each flock contains 2, 3 *or* 10 *circles, in according as* $n = 3$, $n = 5$ *or* $n = 11$.

(vii) Each circle is disjoint with $\frac{n^4-5n^3+9n^2-7n+2}{2(n+1)}$ *circles.*

Proof. (i) Let p be any point of $\mathbf{C}$. By the Proposition 2.3(i) $\mathbf{C}_p$ is a projective plane of order $n-1$. Therefore, there are exactly $(n-1)^2+(n-1)+2$ points in $\mathbf{C}$.

(ii) The total number of lines of $\mathbf{C}_p$ is $(n-1)^2+(n-1)+1$, since $\mathbf{C}_p$ is a projective plane of order $n-1$. Thus, the mention number is the total number of circles on p.

(iii) For each point p of $\mathbf{P}$ the total number of circles on p is $(n-1)^2+(n-1)+1$. Since the total number of points of $\mathcal{C}$ is $(n-1)^2+(n-1)+2$ and each circle of $\mathcal{C}$ contains $n+1$ points, the total number of circles of $\mathcal{C}$ is

$$\left[(n-1)^2+(n-1)+2\right]\left[(n-1)^2+(n-1)+1\right]/(n+1)$$

(iv) Since the total number of points in $\mathcal{C}$ is $(n-1)^2+(n-1)+2$, each circle contains $n+1$ points and each boundle, in $\mathbf{C}$, contains exactly
$\dfrac{(n-1)^2+(n-1)+2-2}{n-1} = n$ circles.

(v) It is trivial, since $k = 0$.

(vi) It is trivial.

(vii) Let c be any circle in $\mathbf{C}$ and the total number of circles which are disjoint with c be an integer number t. If a circle intersects c, then the intersection contains at least two points. The total number of circles intersecting c is $\binom{n+1}{2}(n-1)$, since each boundle contains n circles. Therefore, $t = [(n-1)^2+(n-1)+2][(n-1)^2+(n-1)+1]/(n+1) - [\binom{n+1}{2}(n-1)+1]$
$= \dfrac{n^4-5n^3+9n^2-7n+2}{2(n+1)}$. □

Proposition 2.5. *Let* $\mathbf{C}$ *be a finite circular space in which each circle contains* $n+1$ *points. If* $k = 1$ *and* $n \geq 2$, *then*

(i) The total number of points in $\mathbf{C}$ *is* n^2+1.

(ii) The total number of circles in $\mathbf{C}$ *is* $n(n^2+1)$.

(iii) The total number of circles on each point is n^2+n.

(iv) Each boundle contains $n+1$ *circles.*

(v) Each pencil contains n *circles.*

(vi) Each flock contains $n-1$ *circles.*

(vii) Each circle is tangent to n^2-1 *circles.*

(viii) Each circle is disjoint with $n(n-1)(n-2)/2$ *circles.*

The proof may be found in Dembowski [6].

Proposition 2.6. *Let* $\mathbf{C}$ *be a finite circular space in which each circle contains* $n+1$ *points. If* $k, n \geq 2$ *and* $n \geq 1+\sqrt{1+k}$, *then*

(i) *The total number of circles in* $\mathbf{C}$ *is* $\frac{n+k}{n-1} + 2$.

(ii) *The total number of circles on each point is*

$$[(n+k)(n-1)+1]\,(n+k)/n.$$

(iii) *The total number of circles in* $\mathbf{C}$ *is*

$$\alpha = [(n+k)(n-1)+2]\,[(n+k)(n-1)+1]\,(n+k)/n(n+1).$$

(iv) *Each boundle contains* $n+k$ *circles.*

(v) *Each pencil contains* $n+k-1$ *circles.*

(vi) *Each flock contains* $n+k-3$ *circles.*

(vii) *Each circle is tangent to*

$$\beta = \{[(n+k)(n-1)+1]\,(n+k)/n - [n(n+k-1)+1]\}\,(n+1)$$

circles.

(viii) *Each circle disjoints with* $\alpha - [\beta + \gamma + 1]$ *circles. Where* γ *is the total number of circles meeting* c *in two points, namely*

$$\gamma = \binom{n+1}{2}(n+k-1).$$

Proof. The proof is completely similar to the Proposition 2.4 or the Proposition 2.5. □

Proposition 2.7. *Let* $\mathbf{C}$ *be a finite circular space with* v *points. Then,* $\mathbf{C}$ *is an inversive plane if and only if each boundle* $[p,q]$ *contains exactly* $n+1$ *circles and each circle contains* $n+1$ *points with* $n \in \mathbb{N}$, $n \geq 2$.

Proof. Let $\mathbf{C}$ be an inversive plane, and $[p,q]$ be a boundle. Since $\mathbf{C}_p$ is an affine plane, there is an integer $n, n \geq 2$, such that the boundle $[p,q]$ contains all lines on q in $\mathbf{C}_p$, the total number of lines on each point of $\mathbf{C}_p$ is $n+1$ and each line of $\mathbf{C}_p$ contains n points.

Conversely, let $\mathbf{C}$ be a finite circular space with v points such that each boundle contains $n+1$ circles and each circle contains exactly $n+1$ points, $n \geq 2$. Then the conditions I1 and I2 are trivial, since $\mathbf{C}$ is a circular space. Let c be a circle, p and q be any points such that $q \circ c \phi p$.Then we must show that there is a unique circle c' on p tangent to c at q. But $|[p,q]| = n+1$ and exactly n circles of $[p,q]$ meet c in points different from q by C2, thus one circle of $[p,q]$, say c', is tangent to c at q. □

Proposition 2.8. *Let* $\mathbf{C}$ *be a non-trivial finite circular space with* v *points and* b *circles. Then,* $\mathbf{C}$ *is an Inversive plane iff:*

(i) *There is a possitive integer* n *such that* $b = n.v$.

(ii) *Each circle contains* k *points.*

(iii) $(k-1)^2 = v-1$.

Proof. Let $\mathbf{C}$ be an inversive plane and p be any point of $\mathbf{C}$. It is known that there is an integer $n, n \geq 2$, such that every point is on $n+1$ lines and every line has n points, in the affine plane C_p. Therefore, $v = n^2+1$ and $b = n^3+n = n(n^2+1) = nv$, that is, *(i)* holds. Since every circle of $\mathbf{C}$ contains $k = n+1$ points and $(k-1)^2 = n^2 = v-1$, *(ii)* and *(iii)* also hold.

Conversely, if every circle contains k points, then any boundle of $\mathbf{C}$ contains k circles by (iii). We need that $k = n+1$. Since $(k-1)^2 = v-1$ by (*iii*), $v = (k-1)^2+1$ and

$$b = \frac{\binom{v}{3}}{\binom{k}{3}} = \frac{\binom{(k-1)^2+1}{3}}{\binom{k}{3}} = k^3 - 3k^2 + 4k - 2 = nv.$$

Hence, it is obtained the following equation:

$$k^3 - (n+3)k^2 + (2n+4)k - 2(n+1) = (k-n-1)(k^2-2k+2) = 0.$$

So $k-n-1 = 0$, since $k^2-2k+2 > 0$, that is, $k = n+1$. □

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