On the strong lacunary convergence and strong Cesáro summability of sequences of real-valued functions

A. Gökhan, M. Güngör and Y. Bulut

Abstract. In this paper we introduce the concepts of the strong lacunary convergence and strong Cesáro summabilty of sequences of real-valued functions. We also give the relations between these convergences and pointwise convergence and pointwise statistical convergence.

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1. Introduction

The notion of statistical convergence was introduced by Fast [4] and also independently by Buck [2] and Schoenberg [7] for real and complex number sequences. There is a natural relationship [3] between statistical convergence and strong Cesáro summability:

$$|\sigma_1| = \{x = (x_k) : for\ \ some\, L, \lim_{n\to\infty}(\frac{1}{n}\sum_{k=1}^{n}|x_k - L|) = 0\}.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. There is a strong connection [5] between $|\sigma_1|$ and the sequence space N_θ, which is defined by

$$N_\theta = \{x = (x_k) : for\ \ some\, L, \lim_r(\frac{1}{h_r}\sum_{k\in I_r}|x_k - L|) = 0\}.$$

Gökhan and Güngör [6] defined pointwise statistical convergence by saying that $st - \lim f_k(x) = f(x)$ or $f_k \stackrel{st.}{\to} f$ on S if and only if for every $\varepsilon > 0$,

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$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\le n : |f_k(x)-f(x)|\ge\varepsilon \;\; for\;\; every\; x\in S\;\}\right|=0.$$

It is shown that $\lim f_k(x)=f(x)$ implies $st-\lim f_k(x)=f(x)$ on S.

In this paper we deal with sequences (f_k) whose terms are real-valued functions having a common domain on the real line $\mathbb{R}$.

2. Strong Cesáro Summability and Strong Lacunary Convergence

Definition 2.1. A sequence of functions (f_k) is said to be strongly Cesáro summable if there exists $f(x)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|f_i(x)-f(x)|=0$$

on S. We denote this symbolically by writing $f_k \overset{|\sigma_1|}{\to} f$ or $f_k(x) \overset{|\sigma_1|}{\to} f(x)$ on S.

Theorem 2.1. If a sequence of functions (f_k) is convergent to $f(x)$ on a set S then it is also strongly Cesáro summable to $f(x)$ on the set S.

Proof. The proof is similar to that of Theorem 8.48 of [1]. Therefore we omit it.

But the converse of this theorem is not true. For example, define (f_k) on $\mathbb{R}$ by

$$f_k(x)=\begin{cases} x, & if\;\; k=10^n+i,\; where\; 0\le i\le n,\;\; i\;\; is\;\; even,\; n=1,2,...,\\ -x, & if\;\; k=10^n+i,\; where\;\; 0\le i\le n\;,\; i\;\; is\;\; odd,\; n=1,2,...,\\ 0, & otherwise.\end{cases}$$

Then (f_k) is strongly Cesáro summable to $f(x)=0$ on $\mathbb{R}$. But $\lim\limits_{k\to\infty} f_k(x)$ does not exist.

Theorem 2.2. Let (f_k) and (g_k) be two sequences of functions defined on a set S. If $f_k \overset{|\sigma_1|}{\to} f$ and $g_k \overset{|\sigma_1|}{\to} g$ on S, then $\alpha f_k+\beta g_k \overset{|\sigma_1|}{\to} \alpha f+\beta g$ on S, where $\alpha,\beta\in\mathbb{R}$.

Proof. The proof is clear.

Theorem 2.3. (i) If a sequence (f_k) is strongly Cesáro summable to f on S, then it is pointwise statistically convergent to f on S.

(ii) If a bounded sequence (f_k) is pointwise statistically convergent to f on S, then it is strongly Cesáro summable to f on S.

Proof. (i) Observe that for $f_k \overset{|\sigma_1|}{\to} f$ on S and $\varepsilon>0$, we have that

$$\sum_{k=1}^{n} |f_k(x) - f(x)| \geq |\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \ \ for \ \ every \ \ x \in S \ \}| \, \varepsilon.$$

It follows that if (f_k) is strongly Cesáro summable to f on S, then (f_k) is pointwise statistically convergent to f on S.

(ii) Now suppose that (f_k) is bounded and pointwise statistically convergent to f on S and set $K_x = \sup_k |f_k(x)| + |f(x)|$ for every $x \in S$. Let $\varepsilon > 0$ be given and select N_ε such that

$$\frac{1}{n} \left| \{k \leq n : |f_k(x) - f(x)| \geq \frac{\varepsilon}{2} \ \ for \ \ every \ \ x \in S \ \} \right| < \frac{\varepsilon}{2.K_x}$$

for all $n > N_\varepsilon$ and set

$$L_{n,x} = \{k \leq n : |f_k(x) - f(x)| \geq \frac{\varepsilon}{2} \ \ for \ \ every \ \ x \in S \ \}.$$

Now, for $n > N_\varepsilon$ we have that

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} |f_k(x) - f(x)| &= \frac{1}{n} \left[\sum_{k \in L_{n,x}} |f_k(x) - f(x)| + \sum_{\substack{k \leq n \\ k \notin L_{n,x}}} |f_k(x) - f(x)| \right] \\ &< \frac{1}{n} \frac{n.\varepsilon}{2K_x} K_x + \frac{1}{n} \frac{n.\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $x \in S$. Hence (f_k) is strongly Cesáro summable to f on S.

Definition 2.2. For any lacunary sequence θ, a sequence of functions (f_k) is said to strongly lacunary convergent to f on a set S if there exists $f(x)$ such that

$$\tau_{r,x} = \frac{1}{h_r} \sum_{I_r} |f_k(x) - f(x)| \rightarrow 0$$

for every $x \in S$. We denote this symbolically by writing $f_k \xrightarrow{N_\theta} f$ or $f_k(x) \xrightarrow{N_\theta} f(x)$ on S.

Theorem 2.4. Let (f_k) and (g_k) be two sequences of functions defined on a set S. If $f_k \xrightarrow{N_\theta} f$ on S and $g_k \xrightarrow{N_\theta} g$ on S, then $\alpha f_k + \beta g_k \xrightarrow{N_\theta} \alpha f + \beta g$ on S, where $\alpha, \beta \in \mathbb{R}$.

Proof. The proof is easy. Therefore we omit it.

Lemma 2.1. Let (f_k) be a sequence of functions defined on a set S.

$f_k \overset{|\sigma_1|}{\rightarrow} f$ on S implies $f_k \overset{N_\theta}{\longrightarrow} f$ on S if and only if $\liminf_r q_r > 1$.

Proof **(Sufficiency).** If $\liminf_r q_r > 1$, there exists $\delta > 0$ such that $1+\delta \leq q_r$ for all $r \geq 1$. Then if the sequence (f_k) is strongly Cesáro summable to f on a set S , we can write

$$\begin{aligned} \tau_{r,x} &= \frac{1}{h_r}\sum_{i=1}^{k_r}|f_i(x)-f(x)| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}}|f_i(x)-f(x)| \\ &= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{i=1}^{k_r}|f_i(x)-f(x)|\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}|f_i(x)-f(x)|\right) \end{aligned}$$

for every $x \in S$. Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta} \text{ and } \frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}.$$

The terms $\frac{1}{k_r}\sum\limits_{i=1}^{k_r}|f_i(x)-f(x)|$ and $\frac{1}{k_{r-1}}\sum\limits_{i=1}^{k_{r-1}}|f_i(x)-f(x)|$ both converge to 0 on S, and it follows that $\tau_{r,x}$ converges to 0 for every $x \in$ S, that is, the sequence (f_k) is strongly lacunary convergent to f on S.

(Necessity). Assume that $f_k \overset{|\sigma_1|}{\rightarrow} f$ on S implies $f_k \overset{N_\theta}{\longrightarrow} f$ on S and $\liminf_r q_r = 1$. Since θ is lacunary, we can select a subsequence $(k_{r(j)})$ of θ satisfying $\frac{k_{r(j)}}{k_{r(j)-1}} < 1+\frac{1}{j}$ and $\frac{k_{r(j)-1}}{k_{r(j-1)}} > j$, where $r(j) \geq r(j-1)+2$.

Define (f_i) by

$$f_i(x) = \begin{cases} x, & if \quad i \in I_{r(j)},\ for\ \ some\ \ j=1,2,... \\ 0, & otherwise \end{cases}$$

for every $x \in \mathbb{R}$. Then, for any real-valued function $f(x)$,

$$\frac{1}{h_{r(j)}}\sum_{I_{r(j)}}|f_i(x)-f(x)| = |x-f(x)| \qquad for\ j=1,2,...,$$

and

$$\frac{1}{h_r}\sum_{I_r}|f_i(x)-f(x)| = |f(x)| \qquad for\ \ r \neq r(j)$$

for every $x \in \mathbb{R}$. It follows that (f_i) is not strongly lacunary convergent on $\mathbb{R}$. However, (f_i) is strongly Cesáro summable, since if t is any sufficiently large integer we can find the unique j for which $k_{r(j)-1} < t \leq k_{r(j+1)-1}$ and write

$$\frac{1}{t}\sum_{i=1}^{t}|f_i(x)| \leq |x| \, . \frac{k_{r(j-1)} + h_{r(j)}}{k_{r(j)-1}} \leq \left(\frac{1}{j} + \frac{1}{j}\right)|x| = \frac{2\,|x|}{j}.$$

As $t \to \infty$ it follows that also $j \to \infty$. Hence $f_i \overset{|\sigma_1|}{\to} 0$.

Lemma 2.2. Let (f_k) be a sequence of functions defined on a set S. $f_k \overset{N_\theta}{\to} f$ on S implies $f_k \overset{|\sigma_1|}{\to} f$ on S if and only if $\limsup_r q_r < \infty$.

***Proof* (Sufficiency).** If $\limsup_r q_r < \infty$ there exists $M > 0$ such that $q_r < M$ for all $r \geq 1$. Letting $f_k \overset{N_\theta}{\to} f = 0$ on S and $\varepsilon > 0$ we can then find $R > 0$ and $K > 0$ such that $\sup_{i \geq R} \tau_{i,x} < \varepsilon$ and $\tau_{i,x} < K_x$ for all $i = 1, 2, ...$ and every $x \in S$. Then if t is any integer with $k_{r-1} < t \leq k_r$, where $r > R$, we can write

$$\begin{aligned}
\frac{1}{t}\sum_{i=1}^{t}|f_i(x)| &\leq \frac{1}{k_{r-1}}\sum_{i=1}^{k_r}|f_i(x)| \\
&= \frac{1}{k_{r-1}}\left(\sum_{I_1}|f_i(x)| + \sum_{I_2}|f_i(x)| + ... + \sum_{I_r}|f_i(x)|\right)
\end{aligned}$$

$$= \frac{k_1}{k_{r-1}}\tau_{1,x} + \frac{k_2 - k_1}{k_{r-1}}\tau_{2,x} + ... + \frac{k_R - k_{R-1}}{k_{r-1}}\tau_{R,x}$$

$$+\frac{k_{R+1} - k_R}{k_{r-1}}\tau_{R+1,x} + ... + \frac{k_r - k_{r-1}}{k_{r-1}}\tau_{r,x}$$

$$\leq \left(\sup_{i \geq 1}\tau_{i,x}\right)\frac{k_R}{k_{r-1}} + \left(\sup_{i \geq R}\tau_{i,x}\right)\frac{k_r - k_R}{k_{r-1}}$$

$$< K_x\frac{k_R}{k_{r-1}} + \varepsilon M$$

for every $x \in S$. Since $k_{r-1} \to \infty$ as $t \to \infty$, it follows that $\frac{1}{t}\sum\limits_{i=1}^{t}|f_i(x)| \to 0$ and, consequently, $f_i \overset{|\sigma_1|}{\to} f = 0$ on S.

(Necessity). Assume that $\limsup_r q_r = \infty$. Since θ is lacunary, we can select a subsequence $(k_{r(j)})$ of θ so that $q_{r(j)} > j$ and then define (f_i) by

$$f_i(x) = \begin{cases} x, & if \ \ k_{r(j)-1} < i \leq 2k_{r(j)-1}, \ \ for \ \ some \ \ j = 1, 2, ..., \\ 0, & otherwise \end{cases}$$

for every $x \in \mathbb{R}$. Then

$$\tau_{r(j),x} = \frac{k_{r(j)-1} |x|}{k_{r(j)} - k_{r(j-1)}} < \frac{|x|}{j-1}$$

on $x \in \mathbb{R}$ and, if $r \neq r(j)$, $\tau_{r,x} = 0$ for every $x \in \mathbb{R}$. Thus $f_i \xrightarrow{N_\theta} f = 0$ for every $x \in \mathbb{R}$. Observe next that any sequence which is strongly Cesáro summable consisting of only $f_i(x) = 0$'s or $f_i(x) = x$'s on S has an associated strong limit $f(x)$ which is 0 or x on S. For the sequence (f_i) above, and $i = 1, 2, ..., k_{r(j)}$, $x \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{k_{r(j)}} \sum_i |f_i(x) - x| &\geq \frac{|x|}{k_{r(j)}} \left(k_{r(j)} - 2k_{r(j)-1}\right) \\ &= |x| \left(1 - \frac{2k_{r(j)-1}}{k_{r(j)}}\right) > |x| - \frac{2|x|}{j} \end{aligned}$$

which converges to $|x|$, and, for $i = 1, 2, ..., 2k_{r(j)-1}$,

$$\frac{1}{2k_{r(j)-1}} \sum_i |f_i(x)| \geq \frac{k_{r(j)-1} |x|}{2k_{r(j)-1}} = \frac{|x|}{2},$$

for every $x \in \mathbb{R}$ and it follows that (f_i) does not strongly Cesáro summable on $\mathbb{R}$.

Combining Lemmas 2.1 and 2.2, we have following theorem.

Theorem 2.5. Let θ be a lacunary sequence. Then $f_i \overset{|\sigma_1|}{\rightarrow} f$ and $f_i \xrightarrow{N_\theta} f$ on S if and only if

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty.$$

It is possible for a sequence of functions to have two distinct strong lacunary limits associated with it. An example can be obtained by defining $P(i) = n$, where $n! < i \leq (n+1)!$, and letting

$$f_i(x) = \begin{cases} x, & if \ \ P(i) \ is \ even \\ 0, & if \ P(i) \ is \ odd \end{cases}$$

for $x \in \mathbb{R}$. Then defining $\theta_1 = ((2r)!)$ and $\theta_2 = ((2r+1)!)$, we observe that

$$\frac{1}{h_{r+1}} \sum_{I_{r+1}} |f_i(x)| = \frac{(2r+1)! - (2r)!}{(2(r+1))! - (2r)!} |x| \to 0 \quad (as \ r \to \infty)$$

on $\mathbb{R}$, from which it follows that $f_i \xrightarrow{N_{\theta_1}} f = 0$ on $\mathbb{R}$, and also

$$\frac{1}{h_r}\sum_{I_r}|f_i(x) - x| = \frac{(2r)! - (2r-1)!}{(2r+1)! - (2r-1)!}\,|x| \to 0 \quad (as\ r \to \infty)$$

on $\mathbb{R}$, from which it follows that $f_i(x) \xrightarrow{N_{\theta_2}} f(x) = x$ on $\mathbb{R}$.

Theorem 2.6. If $f_i \overset{|\sigma_1|}{\rightarrow} f$ and $f_i \xrightarrow{N_\theta} g$ on a set S, then $f(x) = g(x)$ for every $x \in S$.

Proof. Let $f_i \overset{|\sigma_1|}{\rightarrow} f$ and $f_i \xrightarrow{N_\theta} g$ on S and suppose that $f(x) \neq g(x)$ for some $x \in S$. We write

$$\begin{aligned}\sigma_{r,x} + \tau_{r,x} &= \frac{1}{h_r}\sum_{I_r}|f_i(x) - f(x)| + \frac{1}{h_r}\sum_{I_r}|f_i(x) - g(x)| \\ &\geq \frac{1}{h_r}\sum_{I_r}|f(x) - g(x)| = |f(x) - g(x)|\end{aligned}$$

for $x \in S$. Since $f_i \xrightarrow{N_\theta} g$ on S, $\tau_{r,x} \to 0$ for $x \in S$. Thus for sufficiently large r we must have

$$\sigma_{r,x} > \frac{1}{2}\,|f(x) - g(x)|$$

for $x \in S$. Observe that

$$\begin{aligned}\frac{1}{k_r}\sum_{i=1}^{k_r}|f_i(x) - f(x)| &\geq \frac{1}{k_r}\sum_{I_r}|f_i(x) - f(x)| = \frac{k_r - k_{r-1}}{k_r}\sigma_{r,x} \\ &= \left(1 - \frac{1}{q_r}\right)\sigma_{r,x} > \frac{1}{2}\left(1 - \frac{1}{q_r}\right)|f(x) - g(x)|\end{aligned}$$

for $x \in S$ and sufficiently large r. Since $f_i \overset{|\sigma_1|}{\rightarrow} f$, the left side converges to $f(x) = 0$ for $x \in S$, so we must have $q_r \to 1$.But this implies, by the proof of Lemma 2.2, that $f_i \xrightarrow{N_\theta} 0 \Rightarrow f_i \overset{|\sigma_1|}{\rightarrow} 0$ on S. Since $(f_i - g) \xrightarrow{N_\theta} 0$ on S, it follows that $(f_i - g) \overset{|\sigma_1|}{\rightarrow} 0$ on S and therefore $\frac{1}{t}\sum_{i=1}^{t}|f_i(x) - g(x)| \to 0$ for $x \in S$. Then we have

$$\frac{1}{t}\sum_{i=1}^{t}|f_i(x) - g(x)| + \frac{1}{t}\sum_{i=1}^{t}|f_i(x) - f(x)| \geq |g(x) - f(x)| > 0$$

for $x \in S$, which yields a contradiction, since both terms on the left converge to 0.

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Authors' address:

A. Gökhan and M. Gügör
Department of Mathematics, Firat University, 23119, Elazig, Turkey
email: agokhan1@firat.edu.tr and mgungor@firat.edu.tr