# On minimizing the norm of linear maps in $C_p$-classes

Messaoud Bounkhel

**Abstract.** In this paper we establish various characterizations of the global minimum of the map $F_\psi : U \to \mathbb{R}^+$ defined by $F_\psi(X) = \|\psi(X)\|_p$, $(1 < p < \infty)$ where $\psi : U \to C_p$ is a map defined by $\psi(X) = S + \phi(X)$ and $\phi : B(H) \to B(H)$ is a linear map, $S \in C_p$, and $U = \{X \in B(H) : \ \phi(X) \in C_p\}$. Further, we apply these results to characterize the operators which are orthogonal to the range of elementary operators.

**M.S.C. 2000**: 47B47, 47A30, 47B20, 47B10.
**Key words**: elementary operators, Schatten $p$-classes, orthogonality, $\varphi$-directional derivative.

## 1 Introduction

Let $E$ be a complex Banach space. We recall ([2]) that $b \in E$ is orthogonal to $a \in E$ (in short $b \perp a$) if for all complex $\lambda$ there holds $\|a + \lambda b\| \geq \|a\|$ . Note that the order is important, that is, if $b$ is orthogonal to $a$, then $a$ need not be orthogonal to $b$. If $E$ is a Hilbert space, then $b \perp a$ is equivalent to $\langle a, b\rangle = 0$, i.e., the orthogonality in the usual sense. Let now $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space $H$ and let $T \in B(H)$ be compact, and let $s_1(X) \geq s_2(X) \geq ... \geq 0$ denote the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator $T$ is said to be belong to the Schatten $p$-classes $C_p$ if

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_i(T)^p\right]^{\frac{1}{p}} = [tr|T|^p]^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

where $tr$ denotes the trace functional. For the general theory of the Schatten $p$-classes the reader is referred to [11]. Recall (see [11]) that the norm $\|\cdot\|$ of the B-space $V$ is said to be Gâteaux differentiable at non-zero elements $x \in V$ if there exists a unique support functional $D_x \in V^*$ such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$ and satisfying

$$\lim_{\mathbb{R}\ni t\to 0} \frac{\|x + ty\| - \|x\|}{t} = Re\, D_x(y),$$

Applied Sciences, Vol.8, 2006, pp. 40-1.

for all $y \in V$. Here $I\!R$ denotes the set of all reals and $Re$ denotes the real part. The Gâteaux differentiability of the norm at $x$ implies that $x$ is a smooth point of the sphere of radius $\|x\|$. It is well known (see [7] and the references therein) that for $1 < p < \infty$, $C_p$ is a uniformly convex Banach space. Therefore every non-zero $T \in C_p$ is a smooth point and in this case the support functional of $T$ is given by

$$D_T(X) = tr\left[\frac{|T|^{p-1}UX^*}{\|T\|_p^{p-1}}\right],$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of $T$. The first result concerning the orthogonality in a Banach space was given by Anderson[1] showing that if $A$ is a normal operator on a Hilbert space $H$, then $AS = SA$ implies that for any bounded linear operator $X$ there holds

$$\|S + AX - XA\| \geq \|S\|. \tag{1.1}$$

This means that the range of the derivation $\delta_A : B(H) \to B(H)$ defined by $\delta_A(X) = AX - XA$ is orthogonal to its kernel. This result has been generalized in two directions: by extending the class of elementary mappings

$$E : B(H) \to B(H);\quad E(X) = \sum_{i=1}^{n} A_i X B_i$$

and

$$\widetilde{E}: B(H) \to B(H);\quad \widetilde{E}\,(X) = \sum_{i=1}^{n} A_i X B_i - X,$$

where $(A_1, A_2, ...A_n)$ and $(B_1, B_2, ...B_n)$ are $n-$ tuples of bounded operators on $H$, and by extending the inequality (1.1) to $C_p$-classes with $1 < p < \infty$ see [4], [8]. The Gâteaux derivative concept was used in [3, 5, 7, 9, 10], in order to characterize those operators which are orthogonal to the range of a derivation. In these papers, the attention was directed to $C_p$-classes for some $p \geq 1$. The main purpose of this note is to characterize the global minimum of the map

$$X \mapsto \|S + \phi(X)\|_{C_p},\ \phi \text{ is a linear map in } B(H),$$

in $C_p$ by using the $\varphi$-directional derivative. These results are then applied to characterize the operators $S \in C_p$ which are orthogonal to the range of elementary operators. It is very interesting to point out that our Theorem 3.3 and its Corollary 3.2 generalize Theorem 1 in [9] and Lemma 2 in [3].

## 2 Preliminaries

**Definition 2.1** Let $(X, \|\cdot\|)$ be an arbitrary Banach space and $F : X \to I\!R$. We define the $\varphi$-directional derivative of $F$ at a point $x \in X$ in direction $y \in X$ by

$$D_\varphi F(x; y) = \lim_{t \to 0^+} \frac{F(x + te^{i\varphi}y) - F(x)}{t}.$$

Note that when $\varphi = 0$ the $\varphi$-directional derivative of $F$ at $x$ in direction $y$ coincides with the usual directional derivative of $F$ at $x$ in a direction $y$ given by

$$DF(x;y) = \lim_{t\to 0^+} \frac{F(x+ty)-F(x)}{t}. \tag{2.1}$$

According to the notation given in [6] we will denote $D_\varphi F(x;y)$ for $F(x) = \|x\|$ by $D_{\varphi,x}(y)$ and for the same function we write $D_x(y)$ for $DF(x;y)$.

**Remark 2.1** In [6] the author used the term $\varphi$-Gâteaux derivative instead of the term "$\varphi$-directional derivative" that we use here. It seems to us that the most appropriate term is the "$\varphi$-directional derivative", because in the classical case when we don't have $\varphi$, as in (2.1) the existence of this limit corresponds to the directional differentiability of $F$ at $x$ in the direction $y$, while the Gâteaux differentiability of $F$ at $x$ corresponds to the existence of the same limit in any direction $y \in E$ and moreover the function $y \mapsto DF(x;y)$ is linear and continuous. We note that the existence of $DF(x;y)$ for any $y \in E$ does not imply the Gâteaux differentiability of $F$ at $x$. Take for example the function $F(x) = \|x\|$. We can easily check that for $x = 0$ one has $DF(x,y) = \|y\|$ for any $y \in E$ but the function $y \mapsto DF(0,y)$ is not linear and so the Gâteaux derivative of $F$ at $x = 0$ does not exist.

We recall (see [8, Proposition 6 ]) that the function $y \mapsto D_{\varphi,x}(y)$ is subadditive and

$$|D_{\varphi,x}(y)| \le \|y\|. \tag{2.2}$$

We end this section by recalling a necessary optimality condition in terms of $\varphi$-directional derivative for a minimization problem.

**Theorem 2.1** ([10]) *Let $(X, \|\cdot\|)$ be an arbitrary Banach space and $F : X \to \mathbb{R}$. If $F$ has a global minimum at $v \in X$, then*

$$\inf_\varphi D_\varphi F(v;y) \ge 0,$$

*for all $y \in X$.*

# 3 Main Results

Let $\phi : B(H) \to B(H)$ be a linear map, that is, $\phi(\alpha X + \beta Y) = \alpha\phi(X) + \beta\phi(Y)$, for all $\alpha, \beta \in \mathbb{C}$ and all $X, Y \in B(H)$, and let $S \in C_p$ $(1 < p < \infty)$. Put

$$\mathbf{U} = \{X \in B(H) : \ \ \phi(X) \in C_p\}.$$

Let $\psi : \mathbf{U} \to C_p$ be defined by

$$\psi(X) = S + \phi(X).$$

Define the function $F_\psi : \mathbf{U} \to \mathbb{R}^+$ by $F_\psi(X) = \|\psi(X)\|_{C_p}$. Now we are ready to prove our first result in $C_p$-classes $(1 < p < \infty)$. It gives a necessary and sufficient optimality condition for minimizing $F_\psi$. The proof of this result follows, with slight modifications, the same lines of the proof of Theorem 3.1 in [10]. For the convenience of the reader we state it.

**Theorem 3.1** *The map $F_\psi$ has a global minimum at $V \in \mathbf{U}$ if and only if*

$$\inf_{\varphi} D_{\varphi,\psi(V)}(\phi(Y)) \geq 0,\ \forall\, Y \in \mathbf{U}. \tag{3.1}$$

*Proof.* For the necessity we have just to combine Theorem 2.1 and the following equality which can be easily checked

$$D_\varphi F_\psi(V,Y) = D_{\varphi,\psi(V)}\left(\phi(Y)\right).$$

Conversely, assume that (3.1) is satisfied. First, observe that

$$\begin{aligned} D_{\varphi,\psi(V)}(e^{i(\pi-\varphi)}\psi(V)) &= \lim_{t\to 0^+} \frac{\left\|\psi(V) + te^{i\varphi}e^{i(\pi-\varphi)}\psi(V)\right\|_{C_p} - \|\psi(V)\|_{C_p}}{t} \\ &= \lim_{t\to 0^+} \frac{\|\psi(V) - t\psi(V)\|_{C_p} - \|\psi(V)\|_{C_p}}{t} \\ &= \|\psi(V)\|_{C_p} \lim_{t\to 0^+} \frac{|1-t|-1}{t} = -\|\psi(V)\|_{C_p}. \end{aligned}$$

¿From this, we have

$$\|\psi(V)\|_{C_p} = -D_{\varphi,\psi(V)}(e^{i(\pi-\varphi)}\psi(V)).$$

Let $Y \in \mathbf{U}$ be arbitrary and put $\widetilde{Y} = -e^{i(\pi-\varphi)}Y + e^{i(\pi-\varphi)}V$. It is easy to see that $\widetilde{Y} \in \mathbf{U}$. Then by (3.1) we have $D_{\varphi,\psi(V)}(\phi(\widetilde{Y})) \geq 0$ and hence by the subadditivity of $D_{\varphi,\psi(V)}(.)$ and the linearity of $\phi$ we get

$$\begin{aligned} \|\psi(V)\|_{C_p} &\leq -D_{\varphi,\psi(V)}(e^{i(\pi-\varphi)}\psi(V)) + D_{\varphi,\psi(V)}(\phi(\widetilde{Y})) \\ &\leq D_{\varphi,\psi(V)}(\phi(\widetilde{Y}) - e^{i(\pi-\varphi)}\psi(V)) \\ &= D_{\varphi,\psi(V)}(-e^{i(\pi-\varphi)}\phi(Y) + e^{i(\pi-\varphi)}\phi(V) - e^{i(\pi-\varphi)}S - e^{i(\pi-\varphi)}\phi(V)) \\ &= D_{\varphi,\psi(V)}(-e^{i(\pi-\varphi)}\psi(Y)). \end{aligned}$$

By using (2.2) and since $Y$ is arbitrary in $\mathbf{U}$, we obtain

$$F_\psi(V) = \|\psi(V)\|_{C_p} \leq D_{\varphi,\psi(V)}(-e^{i(\pi-\varphi)}\psi(Y)) \leq \|\psi(Y)\|_{C_p} = F_\psi(Y), \text{ for all } Y \in \mathbf{U}.$$

Then $F_\psi$ has a global minimum at $V$ on $\mathbf{U}$. □

Let us recall the following result proved in [9] for $C_p$-classes ($1 < p < \infty$).

**Theorem 3.2** ([9]) *Let $X, Y \in C_p$. Then, there holds*

$$D_X(Y) = pRe\left\{tr(|X|^{p-1}U^*Y)\right\},$$

*where $X = U|X|$ is the polar decomposition of $X$.*

The following corollary establishes a characterization of the $\varphi$-directional derivative of the norm in $C_p$-classes ($1 < p < \infty$).

**Corollary 3.1** *Let $X, Y \in C_p$. Then, one has*

$$D_{\varphi,X}(Y) = pRe\left\{e^{i\varphi}tr(|X|^{p-1}U^*Y)\right\},$$

*for all $\varphi$, where $X = U|X|$ is the polar decomposition of $X$.*

*Proof.* Let $X, Y \in C_p$. Put $\widetilde{Y} = e^{i\varphi}Y$. Applying Theorem 3.2 with $\varphi, X$ and $\widetilde{Y}$ we get

$$D_{\varphi,X}(Y) = \lim_{t\to 0^+} \frac{\left\|X + te^{i\varphi}Y\right\|_{C_p} - \|X\|_{C_p}}{t} = \lim_{t\to 0^+} \frac{\left\|X + t\,\widetilde{Y}\right\|_{C_p} - \|X\|_{C_p}}{t} = D_X(\widetilde{Y})$$

$$= pRe\left\{tr(|X|^{p-1}U^*\,\widetilde{Y})\right\} = pRe\left\{e^{i\varphi}tr(|X|^{p-1}U^*Y)\right\}.$$

This completes the proof. □

Now we are going to characterize the global minimum of $F_\psi$ on $C_p$ ($1 < p < \infty$), when $\phi$ is a linear map satisfying the following useful condition:

$$tr(X\phi(Y)) = tr(\phi^*(X)Y), \forall\, X, Y \in C_p, \tag{3.2}$$

where $\phi^*$ is an appropriate conjugate of the linear map $\phi$. We state some examples of $\phi$ and $\phi^*$ which satisfy the above condition (3.2).

1. The elementary operator $E : \mathbf{I} \to \mathbf{I}$ defined by

$$E(X) = \sum_{i=1}^{n} A_i X B_i,$$

where $A_i, B_i \in B(H)$ ($1 \le i \le n$) and $\mathbf{I}$ is a separable ideal of compact operators in $B(H)$ associated with some unitarily invariant norm. In [8, Proposition 8] the author showed that the conjugate operator $E^* : \mathbf{I}^* \to \mathbf{I}^*$ of $E$ has the form

$$E^*(X) = \sum_{i=1}^{n} B_i X A_i,$$

and that the operators $E$ and $E^*$ satisfy the condition (3.2).

2. Using the previous example we can check that the conjugate operator $\widetilde{E}^* : \mathbf{I}^* \to \mathbf{I}^*$ of the elementary operator $\widetilde{E} : \mathbf{I} \to \mathbf{I}$ defined by

$$\widetilde{E}(X) = \sum_{i=1}^{n} A_i X B_i - X,$$

has the form

$$\widetilde{E}^*(X) = \sum_{i=1}^{n} B_i X A_i - X,$$

and that the operators $\widetilde{E}$ and $\widetilde{E}^*$ satisfy the condition (3.2).

Now, we are in position to prove the following theorem.

**Theorem 3.3** *Let $V \in C_p$, and let $\psi(V)$ have the polar decomposition $\psi(V) = U\,|\psi(V)|$. Then $F_\psi$ has a global minimum on $C_p$ at $V$ if and only if $U^*|\psi(V)| \in \ker \phi^*$.*

*Proof.* Assume that $F_\psi$ has a global minimum on $C_p$ at $V$. Then

$$\inf_{\varphi} D_{\varphi,\psi(V)}(\phi(Y)) \geq 0, \tag{3.3}$$

for all $Y \in C_p$. That is,

$$\inf_{\varphi} pRe\left\{e^{i\varphi} tr(|\psi(V)|^{p-1}U^*\phi(Y))\right\} \geq 0, \forall\, Y \in C_p.$$

This implies that

$$tr(|\psi(V)|^{p-1}U^*\phi(Y)) = 0, \forall\, Y \in C_p. \tag{3.4}$$

Let $f \otimes g$, be the rank one operator defined by $x \mapsto \langle x, f\rangle\, g$ where $f, g$ are arbitrary vectors in the Hilbert space $H$. Take $Y = f \otimes g$, since the map $\phi$ satisfies (3.2) one has

$$tr(|\psi(V)|^{p-1}U^*\phi(Y)) = tr(\phi^*(U^*|\psi(V)|^{p-1})Y).$$

Then (3.4) is equivalent to $tr(\phi^*(U^*|\psi(V)|^{p-1})Y) = 0$, for all $Y \in C_p$, or equivalently

$$\left\langle \phi^*(U^*|\psi(V)|^{p-1})g, f\right\rangle = 0, \forall f, g \in H.$$

Thus $\phi^*(U^*|\psi(V)|^{p-1}) = 0$, i.e., $U^*|\psi(V)|^{p-1} \in \ker \phi^*$.

Conversely, let $\varphi$ be arbitrary. If $U^*|\psi(V)|^{p-1} \in \ker \phi^*$, then $e^{i\varphi}U^*|\psi(V)|^{p-1} \in \ker \phi^*$. It is easily seen(using the same arguments above) that

$$Re\left\{e^{i\varphi} tr(U^*|\psi(V)|^{p-1}\phi(Y))\right\} \geq 0, \forall Y \in C_p.$$

Now as $\varphi$ is taken arbitrary, we get (3.3).

We state our first corollary of Theorem 3.3. Let $\phi = \delta_{A,B}$, where $\delta_{A,B} : B(H) \to B(H)$ is the generalized derivation defined by $\delta_{A,B}(X) = AX - XB$.

**Corollary 3.2** *Let $V \in C_p$, and let $\psi(V)$ have the polar decomposition $\psi(V) = U\,|\psi(V)|$. Then $F_\psi$ has a global minimum on $C_p$ at $V$, if and only if $U^*|\psi(V)|^{p-1} \in \ker \delta_{B,A}$.*

*Proof.* It is a direct consequence of Theorem 3.4. □

This result may be reformulated in the following form where the global minimum $V$ does not appear. It characterizes the operators $S$ in $C_p$ which are orthogonal to the range of the derivation $\delta_{A,B}$.

**Theorem 3.4** *Let $S \in C_p$, and let $\psi(S)$ have the polar decomposition $\psi(S) = U\,|\psi(S)|$. Then*

$$\|\psi(X)\|_{C_p} \geq \|\psi(S)\|_{C_p},$$

*for all $X \in C_p$ if and only if $U^*|\psi(S)|^{p-1} \in \ker \delta_{B,A}$.*

As a corollary of this theorem we have

**Corollary 3.3** *Let $S \in C_p \cap \ker \delta_{A,B}$, and let $\psi(S)$ have the polar decomposition $\psi(S) = U\,|\psi(S)|$. Then the two following assertions are equivalent:*

1. $$\|S + (AX - XB)\|_{C_p} \geq \|S\|_{C_p}, \text{ for all } X \in C_p.$$

2. $U^*|S|^{p-1} \in \ker \delta_{B,A}$.

**Remark 3.1** We point out that, thanks to our general results given previously with more general linear maps $\phi$, Theorem 3.4 and its Corollary 3.3 are true for more general classes of operators than $\delta_{A,B}$ like the elementary operators $E(X)$ and $\widetilde{E}\,(X)$.

**Acknowledgement** The author thanks KSU for supporting this work by the Research Center Project No. **Math/1422/22**.

# References

[1] J. Anderson, *On normal derivations*, Proc. Amer. Math. Soc. 38, 1 (1979), 129-135.

[2] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J., 1 (1935), 169-172.

[3] B.P. Duggal, *Range-kernel orthogonality of the elementary operators $X \to \sum_{i=1}^{n} A_i X B_i - X$*, linear Algebra Appl. 337 (2001), 79-86.

[4] B.P. Duggal, *A remark on normal derivations*, Proc. Amer. Math. Soc, 126, 7 (1998), 2047-2052.

[5] D. Keckic, *Orthogonality of the range and the kernel of some elementary operators*, Proc. Amer. Math. Soc. 128, 11 (2000), 3369-3377.

[6] D. Keckic, *Orthogonality in $C_1$ and $C_\infty$-spaces and normal derivations*, J. Op. Theory. to appear.

[7] F. Kittaneh, *Operators that are orthogonal to the range of a derivation*, J. Math. Anal. Appl. 203 (1996), 863-873.

[8] F.Kittaneh, *Normal derivations in norm ideals*, Proc. Amer. Math. Soc. 123, 6 (1995), 1779-1785.

[9] P.J. Maher, *Commutator Approximants*, Proc. Amer. Math. Soc. 115 (1992), 995-1000.

[10] S. Mecheri and M. Bounkhel, *Global minimum and orthogonality in $C_1$-classes*, J. Math. Anal. Appl., 287, 1 (2003), 51-60.

[11] B. Simon, *Trace ideals and their applications*, London Mathematical Society Lecture Notes Series 35, Cambridge University Press, 1979.

[12] Y.Tong, *Kernels of generalized derivations*, Acta. Sci. Math. 54, 102 (1990),159-169.

*Author's address:*

Messaoud Bounkhel
Department of Mathematics, P.O. Box 2455,
Riyadh 11451, Saudi Arabia.
email: bounkhel@ksu.edu.sa