Fuzzy Euclidean ideals

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Abstract. In this paper we define a fuzzy Euclidean ideal of commutative ring with identity and similar fuzzy subsets of different rings. Then we study the intersection and the union of fuzzy Euclidean ideals. We prove some theorems with fuzzy relation. We analyse the relation between fuzzy Euclidean ideals and fuzzy principal ideals. We examine that the fuzzy quotient ideal of R_μ is a fuzzy Euclidean ideal of R_μ. We show that the fuzzy localized subring of a fuzzy Euclidean ideal is also a fuzzy Euclidean ideal.

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1 Introduction

The concept of a fuzzy subset was introduced by Zadeh[14]. Fuzzy subgroup and its important properties were defined and established by Rosenfeld[13]. Then many authors have studied about it. After this time it was necessary to define fuzzy ideal of a ring. The notion of a fuzzy ideal of a ring was introduced by Liu. Malik, Mordeson and Mukherjee have studied fuzzy ideals. The concept of a fuzzy relation on a set was introduced by Zadeh[14]. Bhattacharya and Mukherjee have studied fuzzy relation on groups. Malik and Mordeson [9] studied fuzzy relation on rings.
The aim of this paper is to give a definition for fuzzy Euclidean ideal and to prove some theorems by using it and to investigate the relationships between fuzzy Euclidean ideal and some other fuzzy algebraic structures.
In this paper R is a commutative ring with identity. A fuzzy relation on R is the fuzzy subset of $R \times R$. We define fuzzy Euclidean ideal and similar fuzzy subsets of different rings. We prove that the intersection of any two similar fuzzy Euclidean ideals of R is fuzzy Euclidean ideal of R. But the union of them may not be fuzzy Euclidean ideal of R. But we show that If $\mu \subset \sigma$ then $\mu \cap \sigma$ and $\mu \cup \sigma$ are fuzzy Euclidean ideals of R.
Then we prove some theorems on fuzzy relations by using our definition.That is; if μ and σ are similar fuzzy Euclidean ideals of R, then $\mu \times \sigma$. Conversely, we have that if μ and σ be similar fuzzy subsets of R such that $\mu \times \sigma$ is a fuzzy Euclidean ideal of R, then μ or σ is a fuzzy Euclidean ideal of R. We show that if μ and σ be

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similar fuzzy subsets of R such that $\mu \times \sigma$ is a fuzzy Euclidean ideal of $R \times R$ and $\mu(0) = \sigma(0)$, $\mu(0) \geq \mu(x)$ and $\sigma(0) \geq \sigma(x)$ then bothμ and σ are a fuzzy Euclidean ideals of R. In the fourth part, we prove that if μ is fuzzy Euclidean ideal of R, then $\mu^{'}$ fuzzy quotient ideal determined by μ is fuzzy Euclidean ideal of R_μ. We show that, if μ is a fuzzy Euclidean ideal of R, then the fuzzy localized subring of μ is a fuzzy Euclidean ideal of RS^{-1} We say that μ has the sup property if every nonempty subset of Imμ has a maximal element. We say that fuzzy ideal μ has the M-property if for all $x, y \in R$. We show that any fuzzy Euclidean ideal with sup and M-property is a fuzzy principal ideal.

2 Preliminaries

In this section, we review some definitions and results.

Definition2.1 A fuzzy subset of R is a function $\mu \rightarrow [0, 1]$.

Definition2.2 A fuzzy subset μ of R is called a fuzzy left (right) ideal of R if
(i)$\mu(x - y) \geq min\mu(x), \mu(y)$
(ii) $\mu(xy) \geq (y), (\mu(xy) \geq \mu(x))$.
A fuzzy subset μ of R is called a fuzzy ideal of R if μ is a fuzzy left and fuzzy right ideal of R.

Definition 2.3 If μ is a fuzzy subset of R, then for any $t \in$ Imμ, the set $\mu_t = \{x \in R | \mu(x) \geq t\}$ is called the level subset of R with respect to μ.

Theorem 2.4 *Let μ be fuzzy subset of R. μ is a fuzzy ideal of R if and only if μ_t is an ideal of R for all $t \in Im\mu$.*
Here, if μ is a fuzzy ideal of R, then μ_t is called a level ideal of μ.

Definition 2.5 A fuzzy relation μ on R is the fuzzy subset of $R \times R$.

Definition 2.6 Let μ and σ be fuzzy subsets of R. The cartesian product of μ and σ is $\mu \times \sigma(x, y) = min(\mu(x), \mu(y))$ for all $x, y \in R$

3 Fuzzy Euclidean Ideal

Definition 3.1 A fuzzy ideal μ of R is called a fuzzy Euclidean ideal of R if for all $x, y \in R$, there exists $q \in R$ such that $\mu(x - qy) < \mu(y)$ or $\mu(x - qy) = \mu(y) \in \mu(0), \mu(1))$.
Now we will give relationship between fuzzy Euclidean ideal and Euclidean ideal.

Definition 3.2 ((Agargun, Ersoy);(Fletcher,Agargun) [1,2]) Let Rbe a commutative ring with identity, I is an ideal of R, W is a well ordered set and let $\phi : I \rightarrow W$. I is called an Euclidean ideal of Rif for all $x, y \in\ R$ there exist $q, r \in\ I$ such that $\phi(r) < \phi(y)$ or $r = 0$.

Theorem 3.3 *Let μ be fuzzy ideal of R with sup and M property. If μ is a fuzzy Euclidean ideal of R then μ_t is an Euclidean ideal of R for all $t \in\ Im\mu$.*

Proof. We know that if μ is a fuzzy ideal of R then μ_t is an ideal of for all $t \in\ Im\mu$. So we must show that for all $x, y \in\ \mu_t$ there exist $q, r \in\ \mu_t$ such that $\phi(r) < \phi(y)$ or $r = 0$. Then $\mu(x) \geq\ t$ and $\mu(y) \geq\ t$.

For all $q \in R$, $\mu(qy) \geq t$ and then $\mu(x-qy) \geq t$ implies $r = x - qy \in \mu_t$. Since μ is a fuzzy Euclidean ideal of R with M property, for all $x, y \in \mu_t$, there exist $q, r \in R$ such that $\mu(r) < \mu(y)$ or $\mu(r) = \mu(y) \in \{\mu(0), \mu(1)\}$. If $\mu(r) < \mu(y)$ the proof is complete. If $\mu(r) \geq \mu(y)$ then $\mu(r) = \mu(y) \in \{\mu(0), \mu(1)\}$. From M property, $y \div r$. Then $x = (q+k)y + 0$ and there exists such $k \in R$. Therefore we get $r = 0$. Consequently μ_t is an Euclidean ideal of R for all $t \in \mathrm{Im}\mu$.

Theorem 3.4 *Let μ be fuzzy subset of R with M property. For all $t \in Im\mu$ if μ_t is an Euclidean ideal of R then μ is a fuzzy Euclidean ideal of R.*
Proof : From the hypothesis for all $t \in Im\mu$, μ is a fuzzy ideal of R. So we must show that for all $x, y \in R$, there exist $q, r \in R$ such that $\mu(r) < \mu(y)$ or $\mu(r) = \mu(y) \in \{\mu(0), \mu(1)\}$. If $\mu(r) \geq \mu(y)$ then $r = 0$.
From now on in this section we will deal with intersection and union of fuzzy Euclidean ideals.

Definition 3.5 Let μ and σ be fuzzy subsets of R_1 and R_2 respectively. μ and σ are called similar fuzzy subsets if $min\mu(R_1) = min\sigma(R_1)$ and $max\mu(R_1) = max\sigma(R_2)$.

Theorem 3.6 *If μ and σ are similar fuzzy Euclidean ideals of R then $\mu \cap \sigma$ is fuzzy Euclidean ideal of R.*

Proof. Firstly we must show that $\mu \cap \sigma$ is fuzzy ideal of R. For all $x, y \in R$
$\mu \cap \sigma(x-y) = inf(\mu(x-y), \sigma(x-y))$

$$\geq inf(min\mu(x), \mu(y)), min\sigma(x), \sigma(y))$$

$$= min(inf(\mu(x), \sigma(x)), inf(\mu(y), \sigma(y))$$

$$= min\mu \cap \sigma(x), min\mu \cap \sigma(y).$$

$\mu \cap \sigma(xy) = inf(\mu(xy), \sigma(xy))$

$$\geq inf(max\mu(x), \mu(y)), max\sigma(x), \sigma(y))$$

$$= max(inf(\mu(x), \sigma(x)), inf(\mu(y), \sigma(y))$$

$$= max\mu \cap \sigma(x), max\mu \cap \sigma(y).$$

Secondly we must show that Euclidean condition is satisfied; i.e. For all $x, y \in R$, there exist $q, r \in R$ such that $\mu \cap \sigma(x-qy) < \mu \cap \sigma(y)$ or $\mu \cap \sigma(x-qy) = \mu \cap \sigma(y) \in \{\mu \cap \sigma(0), \mu \cap \sigma(1)\}$.
i If $\mu(x-qy) < \mu(y)$ and $\sigma(x-qy) < \sigma(y)$ then
$\mu \cap \sigma(x-qy) = inf(\mu(x-qy), \sigma(x-qy))\ inf(\mu(y), \sigma(y)) = \mu \cap \sigma(y)$.
ii If $\mu(x-qy) = \mu(y) = \sigma(x-qy) = \sigma(y) = \mu(1)$ then

$$\mu \cap \sigma(x-qy) = inf(\mu(x-qy), \sigma(x-qy))$$

$$= inf(\mu(y), \sigma(y))$$

$$= \mu \cap \sigma(y)$$

$$= \mu \cap \sigma(1).$$

iii If $\mu(x-qy) = \mu(y) = \sigma(x-qy) = \sigma(y) = \mu(0)$ then

$$\begin{aligned}\mu\cap\sigma(x-qy) &= inf(\mu(x-qy),\sigma(x-qy))\\ &= inf(\mu(y),\sigma(y))\\ &= \mu\cap\sigma(y)\\ &= \mu\cap\sigma(0).\end{aligned}$$

iv If $\mu(x-qy)=\mu(y)=\mu(1)$ and $\sigma(x-qy)=\sigma(y)=\sigma(0)$ then

$$\begin{aligned}\mu\cap\sigma(x-qy) &= inf(\mu(x-qy),\sigma(x-qy))\\ &= inf(\mu(y),\sigma(y))\\ &= \mu\cap\sigma(y)\\ &= \mu\cap\sigma(1).\end{aligned}$$

v If $\mu(x-qy)<\mu(y)$ and $\sigma(x-qy)=\sigma(y)=\sigma(1)$ then

$$\begin{aligned}\mu\cap\sigma(x-qy) &= inf(\mu(x-qy),\sigma(x-qy))\\ &= inf(\mu(y),\sigma(y))\\ &= \mu\cap\sigma(y)\\ &= \mu\cap\sigma(1).\end{aligned}$$

vi If $\mu(x-qy)<\mu(y)$ and $\sigma(x-qy)=\sigma(y)=\sigma(0)$ then

$$\begin{aligned}\mu\cap\sigma(x-qy) &= inf(\mu(x-qy),\sigma(x-qy))\\ &= \mu(x-qy))\\ &< \mu(y)\\ &= inf(\mu(y),\sigma(y))\\ &= \mu\cap\sigma(y).\end{aligned}$$

We follow the same procedure for the symmetric cases of (iv), (v) and (vi).
Hence $\mu\cap\sigma$ is a fuzzy Euclidean ideal of R.
However If μ and σ are similar fuzzy Euclidean ideals of R then $\mu\cup\sigma$ may not be a fuzzy Euclidean ideal of R.

Example 3.7 let μ and σ be similar fuzzy subsets of Z such that

$$\mu(x)=\begin{cases}1, & x\in(3)\\ 0, & x\notin(3)\end{cases}$$

It is obviously seen that μ and σ are fuzzy Euclidean ideals of Z, but $\mu\cup\sigma$ is not fuzzy Euclidean ideal of Z. Because

$$\begin{aligned}\mu\cup\sigma(1) &= sup(\mu\cup\sigma(3-2))\\ &= sup(\mu(1),\sigma(1)))\\ &= 0\end{aligned}$$

$$\geq \ min(\mu \cup \sigma(3), \mu \cup \sigma(2)).$$

$$= min(sup(\mu(3), \sigma(3)), sup(\mu(2), \sigma(2)))$$

$$= min(sup(0,1), sup(1,0))$$

$$= 1.$$

Theorem 3.8 *Let μ and σ be fuzzy Euclidean ideals of R such that $\mu \subset \sigma$. Then $\mu \cap \sigma$ and $\mu \cup \sigma$ are fuzzy Euclidean ideals of R.*

Proof. In (Kumar,[6]) Theorem 3.19 show s that for $n = 2$ $\mu \cap \sigma$ and $\mu \cup \sigma$ are fuzzy ideals of R. Then we must show that Euclidean condition is satisfied. For all $x, y \in R$, there exist $q, r \in R$ such that $\mu \cup \sigma(x - qy) = \sigma(x - qy) < \sigma(y) = \mu \cup \sigma(y)$ or

$$\mu \cup \sigma(x - qy) = \sigma(x - qy) = \sigma(y) = \mu \cup \sigma(y)) \in \{\mu \cup \sigma(0), \mu \cup \sigma(1)\}$$

and

$$\mu \cap \sigma(x - qy) = \sigma(x - qy) < \mu(y) = \mu \cap \sigma(y)$$

or

$$\mu \cap \sigma(x - qy) = \mu(x - qy) < \mu(y) = \mu \cap \sigma(y) \in \{\mu \cap \sigma(0), \mu \cap \sigma(1)$$

Therefore $\mu \cap \sigma$ and $\mu \cup \sigma$ are fuzzy Euclidean ideals of R.

Theorem 3.9 *If $\{\mu_n | n \in Z_+\}$ is a collection of fuzzy Euclidean ideals of Rsuch that $\mu_1 \subseteq \mu_2 \subseteq \mu_3 ... \subseteq \mu_n ...$ then $\cup\mu_n$ and $\cap\mu_n$ are fuzzy Euclidean ideals of R.*

Proof. In (Kumar, Dixit and Ajmal [7]) Theorem 3.19 shows that $\cup\mu_n$ and $\cap\mu_n$ are fuzzy ideals of R. Then We must show that Euclidean condition is satisfied. For all $x, y \in R$ there exists $q \in R$ such that

$$\begin{aligned} \mu(x - qy) = \cup\mu_n(x - qy) & \\ &= sup\mu_i(x - qy)|i \in Z^+ \\ &< sup\mu_i(y)|i \in Z^+ \\ &= \cup\mu_n(y). \end{aligned}$$

$$\begin{aligned} \mu(x - qy) = \cup\mu_n(x - qy) & \\ &= sup\{\mu_i(x - qy)|i \in Z^+\} \\ &< sup\{\mu_i(y)|i \in Z^+\} \\ &= \cup\mu_n(y) \in \{\cup\mu_n(0), \cup\mu_n(1)\}. \end{aligned}$$

Therefore $\cup\mu_n$ is fuzzy Euclidean ideal of R. Similary it is shown that $\cap\mu_n$ is fuzzy Euclidean ideal of R.

4 Cartesian Product

Theorem 4.1 *If μ and σ are similar fuzzy Euclidean ideals of R, then $\mu \times \sigma$ is fuzzy Euclidean ideal of $R \times R$.*

Proof. By [9], If μ and σ are fuzzy ideals of R, then $\mu \times \sigma$ is fuzzy ideal of $R \times R$. Take
$(x_1, x_2), (y_1, y_2 \in\ R \times R$. Then there exists $(q_1, q_2) \in R$ such that $\mu \times \sigma(x_1, x_2 - (y_1, y_2)(q_1, q_2) = \mu \times \sigma((y_1, y_2 \in \{\mu \times \sigma(1, 1), \mu \times \sigma(0, 0)\}$.
Here, we consider the following five cases.
If i $\mu(x_1 - y_1q_1) < \mu(y_1$ and $\sigma(x_2 - y_2q_2) < \sigma(y_2)$ then

$$\begin{aligned} \mu \times \sigma(x_1 - y_1q_1, x_2 - y_2q_2) &= min(\mu(x_1 - y_1q_1), \sigma(x_2 - y_2q_2) \\ &< min(\mu(y_1), \sigma(y_2 \\ &= \mu \times \sigma((y_1, y_2). \end{aligned}$$

ii If $\mu(x_1 - (y_1q_1) = \mu(y_1) = \sigma(x_2 - (y_2(q_2) = \sigma(y_2 = \mu(1) = \sigma(1)$ then

$$\begin{aligned} \mu \times \sigma(x_1 - y_1q_1, x_2 - y_2q_2) &= min(\mu(x_1 - y_1q_1), \sigma(x_2 - y_2q_2) \\ &= min(\mu(y_1), \sigma(y_2 \\ &= \mu \times \sigma(1, 1). \end{aligned}$$

iii If $\mu(x_1 - (y_1(q_1) = \mu(y_1) = \sigma(x_2 - (y_2(q_2) = \sigma(y_2 = \mu(0) = \sigma(0)$ then

$$\begin{aligned} \mu \times \sigma((x_1 - y_1q_1, x_2 - y_2q_2) &= min(\mu(x_1 - y_1q_1), \sigma(x_2 - y_2q_2) \\ &= min(\mu(y_1), \sigma(y_2 \\ &= \mu \times \sigma(0, 0). \end{aligned}$$

iv If $\mu(x_1 - y_1q_1) = \mu(y_1) = \mu(1)$ and $\sigma(x_2 - y_2q_2) = \sigma(y_2) = \sigma(0)$ then

$$\begin{aligned} \mu \times \sigma(x_1 - y_1q_1, x_2) - y_2q_2) &= \min(\mu(x_1 - y_1q_1), \sigma(x_2 - y_2q_2) \\ &= \mu((x_1 - y_1q_1) \\ &= \mu(y_1 \\ &= \min(\mu(y_1), \sigma(y_2 \\ &= \mu \times \sigma(y_1, y_2) \\ &= \mu \times \sigma(1, 1). \end{aligned}$$

v If $\mu(x_1 - (y_1(q_1) < \mu(y_1)$ and $\sigma(x_2 - (y_2(q_2) = \sigma(y_2 = \sigma(1)$ then

$$\begin{aligned} \mu \times \sigma((x_1 - y_1q_1, x_2 - y_2q_2) &= \min(\mu(x_1 - y_1q_1), \sigma(x_2 - y_2q_2) \\ &= \sigma((x_2 - y_2q_2) \\ &= \sigma(y_2 \\ &= \min(\mu(y_1), \sigma(y_2 \\ &= \mu \times \sigma(y_1, y_2) \\ &= \mu \times \sigma(1, 1). \end{aligned}$$

vi If $\mu(x_1 - (y_1(q_1) < \mu(y_1)$ and $\sigma(x_2 - (y_2(q_2) = \sigma(y_2 = \sigma(0)$ then

$$\begin{aligned}\mu \times \sigma((x_1 - y_1q_1, x_2 - y_2q_2) &= = min(\mu(x_1 - y_1q_1), \sigma(x_2 - y_2q_2) \\ &= \mu((x_1 - y_1q_1) < \mu(y_1 \\ &= = min(\mu(y_1), \sigma(y_2 \\ &= = \mu \times \sigma(y_1, y_2).\end{aligned}$$

We follow the same procedure for the symmetric cases of (v) and **(vi)**. Therefore for all $(x_1, x_2), (y_1, y_2) \in R \times R$ there exists $(q_1, q_2) \in R \times R$, such that $\mu \times \sigma((x_1, x_2) - (y_1, y_2)(q_1, q_2) = \mu \times \sigma((y_1, y_2) \in \{\mu \times \sigma(1, 1), \mu \times \sigma(0, 0)\}$.
Hence $\mu \times \sigma$ a is fuzzy Euclidean ideal of $R \times R$.

Theorem 4.2 *Let μ and σ be similar fuzzy subsets of R such that $\mu \times \sigma$ is a fuzzy Euclidean ideal of $R \times R$. Then μ or σ is a fuzzy Euclidean ideal of R.*

Proof. By (Malik, Mordeson [9]), we know that if $\mu \times \sigma$ is a fuzzy ideal of $R \times R$ then μ or σ is a fuzzy ideal of R and also $\mu(x) \leq \mu(0)$ for all $x \in R$ or $\sigma(x) \leq \sigma(0)$ for all $x \in R$
If $\mu(x) \leq \mu(0)$ then we have $\mu(x) \leq \sigma(0)$ or $\sigma(x) \leq \sigma(0)$ for all $x \in R$
Take $x, y \in R$ and consider the following two cases;
Case I: If $\mu(x) \leq \mu(0)$ then $\mu \times \sigma(x, 0) = \mu(x)$ for all $x \in R$. Since $\mu \times \sigma$ is fuzzy Euclidean ideal, for all $(x, 0), (y, 0) \in R \times R$, there exists $(q_1, q_2) \in R \times R$ such that $\mu \times \sigma((x_1, 0) - (y_1, 0)(q_1, q_2)) < \mu \times \sigma((y_1, 0)$ or $\mu \times \sigma\{(x_1, 0) - (y_1, 0)(q_1, q_2))\} = \mu \times \sigma((y_1, 0) \in \{\mu \times \sigma(1, 1), \mu \times \sigma(0, 0)\}$.
Then we get for all $x, y \in R$ there exists $q_1 \in R$ such that $\mu \times \sigma((x_1, 0) - (y_1, 0)(q_1, q_2) < \mu(y_1) = \mu \times \sigma((y_1, 0)$ or $\mu \times \sigma((x_1, 0) - (y_1, 0)(q_1, q_2) = \mu \times \sigma((y_1, 0) = \mu(y_1) \in \{\mu(1), \mu(0)\}$.
Case II: Suppose $\sigma(x) \leq \sigma(0)$ for all$x \in R$ and case I does not hold. Then there exists $a \in R$ such that $\mu(a) > \sigma(0)$ Therefore $\mu(0) \geq \mu(a) > \sigma(0)$ Hence
$\sigma(x) = min(\mu(0), \sigma(x)) = \mu \times \sigma(0, x)$ for all $x \in\ R$.
Since $\mu \times \sigma$ is fuzzy Euclidean ideal, for all $(0, x), (0, y) \in R$ there exists $(q_1, q_2 \in R \times R$ such that $\mu \times \sigma((0, x_2) - (0, y_2)(q_1, q_2)) < \mu \times \sigma(0, y_2)$ or $\mu \times \sigma((0, x_2) - (0, y_2)(q_1, q_2)) = \mu \times \sigma((0, y_2) \in \mu \times \sigma(0, 0), \mu \times \sigma(1, 1)$.
Then we get for all$x, y \in R$ there exists $q_2 \in R$ such that $\sigma((x_2 - y_2q_2 = \mu \times \sigma(0, (x_2 - y_2q_2) < \mu \times \sigma(0, y_2) = \sigma(y_2$ or $\sigma((x_2 - y_2q_2 = \mu \times \sigma(0, (x_2 - y_2q_2) = \mu \times \sigma(0, y_2) = \sigma(y_2 \in \{\sigma(0), \sigma(1)\}$.
Hence either μ or σ is a fuzzy Euclidean ideal of R.

Theorem 4.3 *let μ and σ be similar fuzzy subsets of R such that $\mu \times \sigma$ is a fuzzy Euclidean ideal of $R \times R$.If $\mu(0) = \sigma(0), \mu(0) \geq \mu(x)$, and $\sigma(0) \geq \sigma(x)$ for all $x \in R$ then both μ and σ are fuzzy Euclidean ideals of R.*

Proof. By (Malik, Mordeson[9]), we know that in these conditions μ and σ are fuzzy ideals of R. So we need to show that for all $x, y \in R$, there exists $q_1q_2 \in R$ such that $\mu(x - yq_1) < \mu(y)$ or $\mu(x - yq_1) = \mu(y) \in \{\mu(0), \mu(1)\}$ and $\sigma(x - yq_2) < \sigma(y)$ or $\sigma(x - yq_2) = \sigma(y) \in \{\sigma(0), \sigma(1)\}$.
We have $\mu(x) = \mu \times \sigma(x, 0)$ and Since $\sigma(x) = \mu \times \sigma(0, x)$ for all $x \in R$. Since $\mu \times \sigma$ is fuzzy Euclidean ideal, for all $(x, 0), (y, 0) \in R \times R$, there exists $(q_1, q_2) \in R \times R$ such that $\mu \times \sigma((x - yq_1, 0)) < \mu \times \sigma(y, 0)$ or $\mu \times \sigma((x, 0) - (y, 0)(q_1, q_2)) = \mu \times \sigma(y, 0) \in \{\mu \times \sigma(1, 1), \mu \times \sigma(0, 0)\}$.

Then we get for all $x, y \in R$ there exists $q_1 \in R$ such that $\mu \times \sigma(x - yq_1, 0) = \mu(x - yq_1) < \mu(y) = \mu \times \sigma(y, 0)$ or $\mu \times \sigma((x - yq_1, 0) = \mu(x - yq_1) = \mu(y) = \mu \times \sigma(y, 0) \in \{\mu(1), \mu(0)\}$.
Since $\mu \times \sigma$ is fuzzy Euclidean ideal, for all $(0, x), (0, y) \in R \times R$, there exists $(q_1, q_2) \in R \times R$ such that $\mu \times \sigma(0, x - yq_2) < \mu \times \sigma((0, y)$ or $\mu \times \sigma((0, x) - (0, y)(q_1, q_2) = \mu \times \sigma(0, y) \in \{\mu \times \sigma(1, 1), \mu \times \sigma(0, 0)\}$.
Then we get for all $x, y \in R$ there exists $q_2 \in R$ such that $\sigma(x - yq_2) = \mu \times \sigma(0, x - yq_2) < \mu \times \sigma(0, y) = \sigma(y)$ or $\sigma(x - yq_2) = \mu \times \sigma(0, x - yq_2) = \mu \times \sigma(0, y) = \sigma(y) \in \{\sigma(1), \sigma(0)\}$.
Hence, both μ and σ are fuzzy Euclidean ideals of R.

Corollary 4.4 *Let σ be a fuzzy subset of R. Then $\mu_{sigma} = \sigma \times \sigma$ is a fuzzy Euclidean ideal of $R \times R$ if and only if σ is a fuzzy Euclidean ideal of R.*

Proof. It is immediate from Theorem 4.1 and Theorem 4.2.

Example 4.5 Let μ be a fuzzy subset of Z such that

$$\mu(x) = \begin{cases} 1, & x = 0 \\ 0.5, & x \in (2) \\ 0, & x \notin (2) \end{cases}$$

then μ is a fuzzy ideal of Z but not fuzzy Euclidean. Because for example; take $x = 6$ and $y = 4$. Here, we can not find any q integer such that $\mu(x - qy) < \mu(y)$ or $\mu(x - qy) = \mu(y) \in \{\mu(0), \mu(1)\}$. Therefore μ is not fuzzy Euclidean ideal of Z.

Example 4.6 let μ be a fuzzy subset of Z such that

$$\mu(x) = \begin{cases} 0.5, & x \in (3) \\ 0.4, & x \notin (3) \end{cases}$$

Then μ is a fuzzy Euclidean ideal of Z.

5 Fuzzy Quotient ideal, Fuzzy localized Subrings and Fuzzy Principal ideal

Definition 5.1(Alkhamees, Mordeson,[4])Let μ be any fuzzy ideal of a ring R and let the fuzzy subset μ_x^* of R defined by $\mu_x^*(r) = \mu(r - x)$ for all $x \in R$ is termed as the fuzzy coset determined by x and μ.

Theorem 5.2 (Alkhamees, Mordeson,[4]) *Let μ be any fuzzy ideal of a ring R. Then R_μ, the set of all fuzzy cosets of μ in R, is a ring under the binary compositions $\mu_x^* + \mu_y^* = \mu_{x+y}^*$ and $\mu_x^* \mu_y^* = \mu_{xy}^*$ for all $x, y \in R$.*

Definition 5.3(Alkhamees, Mordeson,[4]) If μ is any fuzzy ideal of a ring R, then the fuzzy ideal $\mu^{'}$ of defined by $\mu^{'}(\mu_x^*) = \mu(x)$ for all $x \in R$ is called the fuzzy quotient ideal determined by μ.

Theorem 5.4 *Suppose μ is a fuzzy Euclidean ideal of R and $\mu^{'}$ is the fuzzy quotient ideal determined by μ. If μ is fuzzy Euclidean ideal of R then $\mu^{'}$ is fuzzy Euclidean ideal of R_μ.*

Proof. $\mu^{'} : R_\mu \to [0,1]$ is fuzzy quotient ideal. Since μ is fuzzy Euclidean ideal of R, for all $x, y \in R$ there exists $q \in R$ such that $\mu(x-qy) < \mu(y)$ or $\mu(x-qy) = \mu(y) \in \mu(0), \mu(1)$.
Then $\mu^{'}(\mu^*_{x-qy}) = \mu^{'}(\mu^*_x - \mu^*_{qy}) = \mu^{'}(\mu^*_x - \mu^*_q\mu^*_y)$.

If $\mu(x-qy) < \mu(y)$, then $\mu^{'}(\mu^*_{x-qy}) = \mu(x-qy) < \mu(y) = \mu^{'}(\mu_y)$.
If $\mu(x-qy) = \mu(y) = \mu(1)$ then
$\mu^{'}(\mu^*_{x-qy}) = \mu(x-qy) = \mu(y) = \mu^{'}(\mu_y) = min\mu^{'}(1_{R_\mu})$ and
If $\mu(x-qy) = \mu(y) = \mu(0)$ then
$\mu^{'}(\mu^*_{x-qy}) = \mu(x-qy) = \mu(y) = \mu^{'}(\mu_y) = min\mu^{'}(0_{R_\mu})$

Therefore $\mu^{'}$ is fuzzy Euclidean ideal of R_μ .
Throughout the remainder of the paper, S denotes a multiplicative closed set of R such that $0 \in S$ and which is saturated. Let RS^{-1} denote the corresponding ring fractions.

We have the following definition.

Definition 5.5 (Alkhamees, Mordeson,[5]) Assume that μ and $\mu^{'}$ are fuzzy ideals of R and RS^{-1} respectively. Then $\mu^{'}$ is called the fuzzy localized subring of μ in RS^{-1} if $Im(\mu) = Im(\mu^{'})$ and $\mu^{'}_t = \mu_t S^{-1}$ for all $t \in Im(\mu)$.

Theorem 5.6 *If μ is a fuzzy Euclidean ideal of R, then $\mu^{'}$ is a fuzzy Euclidean ideal of RS^{-1}.*

Proof. First we show that $\mu(a) = \mu^{'}(a/s)$ for all $a/s \in RS^{-1}$ Suppose $\mu(a/s) = t$. Then we have

$$a/s \in \mu'_t \Rightarrow a \in \mu_t \Rightarrow \mu(a) \geq t \Rightarrow \mu'(a/s) \leq \mu(a) \quad (1)$$

Conversely, suppose $\mu(a) = t$. Then we have $a \in \mu_t \Rightarrow \frac{a}{s} \in \mu_t S^{-1} = \mu^{'}_t \Rightarrow$

$$(2)\mu^{'}(a/s) \geq t \Rightarrow \mu(a) \leq \mu^{'}(a/s).$$

Therefore from (1) and (2) we get $\mu(a) = \mu^{'}(a/s)$. Take $a/s, b/s_1 \in RS^{-1}$. Since μ is a fuzzy Euclidean ideal of R, there exist $q \in R$ such that $\mu(a-qb) < \mu(b)$ or $\mu(a-qb) = \mu(b) \in \{\mu(1), \mu(0)\}$. Then we have in RS^{-1} $\mu^{'}(a/s) - (qs_1/s)b/s_1)$ $= \mu^{'}(a-qb)/s = \mu(a-qb) < \mu(b) = \mu^{'}(b/s_1)$. or $\mu^{'}(a/s) - qs_1/sb/s_1) = \mu^{'}(a-qb)/s$ $= \mu(a-qb) = \mu(b) = \mu^{'}(b/s_1) \in \{\mu^{'}(1_{RS^{-1}}, \mu^{'}(0_{RS^{-1}}\}$.
Hence $\mu^{'}$ is a fuzzy Euclidean ideal of RS^{-1}

Definition 5.7(Alkhamees, Mordeson,[5]) A fuzzy ideal of R is called fuzzy principal if it has a minimal generating set Im such that if x_s and y_t are in Im with $s > t$, then $x = r_s y$ for some $r_s \in R$. Such a generating set is called a principal generating set for μ.

Theorem 5.8 (Alkhamees, Mordeson,[5]) *Let μ be fuzzy ideal of R. Suppose that μ has the sup property. Then μ is a fuzzy principal if and only if every level ideal μ_t of μ is principal.*

Theorem 5.9 *If μ is fuzzy Euclidean ideal of R with the sup property and M-property, then μ is a fuzzy principal ideal.*

Proof. By Theorem 5.8 it is enough to show that level ideal μ_t is principal. Suppose $\mu(y)$ is a minimum element in $Im(\mu)$ for all $y \in \mu_t$.

Since $y \in \mu_t$,

(1) $$(y) \subset \mu_t.$$

Since $x - qy \in \mu_t$ and from choising of $\mu(y)$ $\mu(x-qy) < \mu(y)$. Since μ has M-property we have $y \div r$. Therefore $x = y(q+k)$, there exist $k \in R$. So

(2) $$x \in (y) \Rightarrow \mu_t \subset (y).$$

From (1) and (2) $\mu_t = (y)$. Therefore, μ_t is principal and by Theorem 4.8 and so μ is a fuzzy principal ideal.

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