The investigation of some topologies on finite sets

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Abstract. The topologies on the finite sets are considered. The problems, connected with the classification of the topologies, with the calculation of the number of the topologies, which consist of the given number of the elements $m = 3, ..., 7$, are solved. The notion of a matreshka is introduced. The initial problem is reduced to the calculation of the number of the matreshkas. The obtained theoretical results are verified by experiments. The experiments were carried out on a computer.

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1 Statement of the problem

Let the set T of n elements is given. The topological structure on it can be constructed in various ways. The problem, connected with the calculation of the topologies on the set T, is not solved yet. The problem, connected with the finding of the number of the topologies, which have the given number of the elements (from 2 to 7) is posed in this article. Since the topology is the subsystem of the system of all the subsets of the set T, then it's inconveniently to select the topologies for a great n directly, even with the help of a computer. As is known, the number of all the subsets of the set T is equal to 2^n.

Let us divide all the topologies into the classes. We shall associate all the topologies, which consist of $m + 2$ elements with m-class $m = 0, 1, ...2^n - 2$. Namely, all the topologies of m-class contain m nontrivial elements. In particular, the trivial topology and the discrete topology are considered to be the topologies of 0-class and $2^n - 2$-class respectively.

Let us designate by σ_i the number of all the topologies in i-class. σ_i, $i = \overline{1, 5}$ are calculated in the article.

2 The complete and noncomplete topologies

We shall divide all the topologies in each class into two subclasses - the complete and the noncomplete ones. The topology on the set T is called the complete one, if the union of its nontrivial elements gives all set T. Otherwise we shall call this topology

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the noncomplete. The following lemma allows to find the properties of the complete topologies.

Lemma *If the topology is complete, then there exists at least one pair of its nontrivial elements A, B, the union of which gives set T.*

Proof. The topology τ is complete. Then $\bigcup_{A_i \in \tau} = A_1 \cup A_2 \cup ... \cup A_k = T$. Let us designate by $B_1 = A_1 \cup A_2 \cup ... \cup A_{k-1}$ (according to the definition $B_1 \in \tau$), then $T = B_1 \cup A_k$. If $B_1 \neq T$, then the lemma is proved. If $B_1 = T$, then $T = A_1 \cup A_2 \cup ... \cup A_{k-1}$. Obviously, we shall obtain $T = A_1 \cup A_2$ on the some step of the recursion, at least on $(k-1)$ step.

□

3 Matreshkas

We shall call the matreshka the system of the subsets, which contains $\varnothing$ and T, where all the elements are ordered with respect to the inclusion. The number of nontrivial elements in the matreshka is called its length. Obviously, the matreshka is considered to be the topology. The matreshka is called the dense one if the addition of at least one element gives the system of the subsets, which is not considered to be the matreshka. Otherwise, the matreshka is called nondense. The dense matreshkas on n-element set T belong to $(n-1)$-class and have the form

$$\tau = \{\varnothing, a_{i_1}, a_{i_1}a_{i_2}, ..., a_{i_1}...a_{i_{n-1}}\}.$$

Let us designate by $S(n,m)$ the number of the different matreshkas of m length on n-element set.

Theorem 1. *The number of the matreshkas in m-class is related to the number of the matreshkas in $(m-1)$-class by the following recurrent formula:*

$$S(n,m) = \sum_{k=m}^{n-1} C_n^k S(k, m-1). \tag{3.1}$$

Proof. We shall use the induction on m. Let us designate by $S(n, m, p_1, p_2, ..., p_m)$ the number of the matreshkas of the length m on n-element set, i-th element of these matreshkas contains p_i elements $(p_1 > p_2 > ... > p_m)$. The formula

$$S(n, m, p_1, p_2, ..., p_m) = C_n^{p_1} C_{p_1}^{p_2} ... C_{p_{m-1}}^{p_m}$$

takes place. Then

$$S(n,m) = \sum_{n-1 \geq p_1 > p_2 > ... > p_m \geq 1} S(n, m, p_1, p_2, ..., p_m).$$

If $m = 1$

$$S(n,1) = \sum_{n-1 \geq p_1 \geq 1} S(n, 1, p_1) = \sum_{p_1=1}^{n-1} C_n^{p_1}.$$

If $m = 2$

$$S(n,2) = \sum_{n-1 \geq p_1 > p_2 \geq 1} S(n,2,p_1,p_2) = \sum_{n-1 \geq p_1 > p_2 \geq 1} C_n^{p_1} C_{p_1}^{p_2} =$$

$$= \sum_{p_1=2}^{n-1} \sum_{p_2=1}^{p_1-1} C_n^{p_1} C_{p_1}^{p_2} = \sum_{p_1=2}^{n-1} C_n^{p_1} \sum_{p_2=1}^{p_1-1} C_{p_1}^{p_2} = \sum_{p_1=2}^{n-1} C_n^{p_1} S(p_1,1)$$

We shall assume that for the matreshkas of the length $m-1$ the formula

$$S(n,m-1) = \sum_{k=m-1}^{n-1} C_n^k S(k,m-2) =$$

$$\sum_{p_1=m}^{n-1} C_n^{p_1} \sum_{p_2=m-1}^{p_1-1} C_{p_1}^{p_2} \ldots \sum_{p_{m-1}=1}^{p_{m-2}-1} C_{p_{m-2}}^{p_{m-1}}$$

takes place. Let us prove it for the matreshkas of the length m.

$$S(n,m) = \sum_{n-1 \geq p_1 > p_2 > \ldots > p_m \geq 1} S(n,m,p_1,p_2,\ldots,p_m) =$$

$$= \sum_{p_1=m}^{n-1} C_n^{p_1} \sum_{p_2=m-1}^{p_1-1} C_{p_1}^{p_2} \ldots \sum_{p_m=1}^{p_{m-1}-1} C_{p_{m-1}}^{p_m} = \sum_{p_1=m}^{n-1} C_n^{p_1} S(p_1,m-1).$$

The theorem is proved.

□

Theorem 2. *Formula*

$$S(n,m) = (-1)^{m+1} \sum_{k=1}^{m+1} (-1)^k C_{m+1}^k k^n \qquad (3.2)$$

takes place.

It is possible to prove this theorem using the induction on m, and the formula (3.1). Also it is necessary to note that $S(n,1) = 2^n - 2$.

In the case of the dense matreshkas the formula (3.2) gives $S(n,n-1) = n!$ Let us note, that if $m > n-1$, then $S(n,m)$ is equal to zero.

4 The calculation of the number of the topologies of m-class ($m = \overline{1,3}$)

The topologies of the 1-class have the form $\tau = \{\varnothing, A, T\}$. Then

$$\sigma_1 = 2^n - 2.$$

The topologies of the 2-class have the form $\tau = \{\varnothing, A, B, T\}$. The equality $B = T \setminus A$ takes place in the complete topologies ($A \cup B = T$). The topology

$\tau = \{\varnothing, A, T \setminus A, T\}$ is put in correspondence with the matreshkas $\{\varnothing, A, T\}$ or $\{\varnothing, T \setminus A, T\}$. Then

$$\sigma_{2,1} = \frac{S(n,1)}{2} = 2^{n-1} - 1.$$

The equality $A \cup B = B$ takes place in the noncomplete topologies ($A \cup B \neq T$), under the condition of $|A| \leq |B|$. These topologies are considered to be the matreshkas of the length 2. That is why

$$\sigma_{2,2} = S(n,2) = 3^n - 3 \cdot 2^n + 3.$$

Then

$$\sigma_2 = \sigma_{2,1} + \sigma_{2,2} = \frac{S(n,1)}{2} + S(n,2).$$

The topologies of the 3-class have the form $\tau = \{\varnothing, A, B, C, T\}$. The equality $B \cap C = A$ takes place in the complete topologies ($B \cup C = T$). These topologies have the form $\{\varnothing, A, B, (T \setminus B) \cup A, T\}$ or $\{\varnothing, A, C, (T \setminus C) \cup A, T\}$, because they can be determined with the help of the matreshka $\{\varnothing, A, B, T\}$, or with the help of the matreshka $\{\varnothing, A, C, T\}$, then the number of the topologies in this subclass will be

$$\sigma_{3,1} = \frac{S(n,2)}{2} = \frac{1}{2}\left(3^n - 3 \cdot 2^n + 3\right).$$

Let us consider the noncomplete topologies. For these topologies $A \cup B = C$. Only two variants are possible: $A \cap B = \varnothing$ or $A \cap B = B$, that is we have two more subclasses of the 3-class - the second and the third respectively. Each topology of the second subclass is defined by the matreshka $\{\varnothing, A, C, T\}$ of the length 2 and has the form $\{\varnothing, A, C \setminus A, C, T\}$, thus

$$\sigma_{3,2} = \frac{S(n,2)}{2} = \frac{1}{2}(3^n - 3 \cdot 2^n + 3).$$

Since in the third subclass $A \subset B \subset C$, then its topologies are the matreshkas of the length 3. According to the formula (3.2), we shall obtain

$$\sigma_{3,3} = S(n,3) = 4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4.$$

Thus, in 3-class of the topologies one has

$$\sigma_3 = \sigma_{3,1} + \sigma_{3,2} + \sigma_{3,3} = S(n,2) + S(n,3).$$

5 The topologies of the 4-and 5-classes

The data on the number of the topologies in these classes are given in the following tables. The obtained results were checked up with the help of the programs in computer algebra system GAP (`http://www.gap-system.org`).

	The subclasses	The corresponding matreshkas and the other open sets	The number of the topologies in the subclass
1	$C \cup D = T, C \cap D = B,$ $A \cup B = C, (A \cup B = \varnothing)$	$\{\varnothing, B, C, T\},$ $A = C \setminus B,$ $D = (T \setminus C) \cup B$	$\sigma_{4,1} = S(n,2)$
2	$C \cup D = T, C \cap D = B,$ $A \cup B = B$	$\{\varnothing, A, B, C, T\},$ $D = (T \setminus C) \cup B$	$\sigma_{4,2} = \frac{S(n,3)}{2}$
3	$C \cup D \neq T, B \cup C = D,$ $B \cap C = A$	$\{\varnothing, A, B, D, T\},$ $C = (D \setminus B) \cup A$	$\sigma_{4,3} = \frac{S(n,3)}{2}$
4	$C \cup D \neq T, C \subset D,$ $A \cup B = C, A \cap B = \varnothing$	$\{\varnothing, A, C, D, T\},$ $B = C \setminus A$	$\sigma_{4,4} = \frac{S(n,3)}{2}$
5	$C \cup D \neq T,$ $A \subset B \subset C \subset D$	$\{\varnothing, A, B, C, D, T\}$	$\sigma_{4,5} = S(n,4)$

$$\tau = \{\varnothing, A, B, C, D, T\}; \qquad \sigma_4 = S(n,2) + \frac{3}{2}S(n,3) + S(n,4).$$

Table l. 4-class

	The subclasses	The corresponding matreshkas and the other open sets	The number of the topologies in the subclass
1	$D \cup E = T, D \cap E = C,$ $A \subset B \subset C$	$\{\varnothing, A, B, C, D, T\},$ $E = (T \setminus D) \cup C$	$\sigma_{5,1} = \frac{S(n,4)}{2}$
2	$D \cup E = T, D \cap E = C,$ $A \cup B = C,\ A \cap B = \varnothing$	$\{\varnothing, A, C, D, T\},$ $B = C \setminus A,$ $E = (T \setminus D) \cup C$	$\sigma_{5,2} = \frac{S(n,3)}{4}$
3	$D \cup E = T, D \cap E = C,$ $B \subset C, B \cap C = A, A \subset C$	$\{\varnothing, A, C, D, T\},$ $B = (D \setminus C) \cup A,$ $E = (T \setminus D) \cup C$	$\sigma_{5,3} = S(n,3)$
4	$D \cup E \neq T, C \cup D = E,$ $C \cap D = A, A \cup B = C,$ $A \cap B = \varnothing$	$\{\varnothing, A, C, E, T\},$ $B = C \setminus A,$ $D = (E \setminus C) \cup A$	$\sigma_{5,4} = S(n,3)$
5	$D \cup E \neq T, C \cup D = E,$ $C \cap D = B, A \cap B = A$	$\{\varnothing, A, B, C, E, T\},$ $D = (E \setminus C) \cup B$	$\sigma_{5,5} = \frac{S(n,4)}{2}$
6	$D \cup E \neq T, D \subset E,$ $B \cup C = D, B \cap C = A$	$\{\varnothing, A, C, D, E, T\},$ $B = (D \setminus C) \cup A$	$\sigma_{5,6} = \frac{S(n,4)}{2}$
7	$D \cup E \neq T, C \subset D \subset E,$ $A \cup B = C, A \cap B = \varnothing$	$\{\varnothing, A, C, D, E, T\},$ $B = C \setminus A$	$\sigma_{5,7} = \frac{S(n,4)}{2}$
8	$D \cup E \neq T,$ $A \subset B \subset C \subset D \subset E$	$\{\varnothing, A, B, C, D, E, T\}$	$\sigma_{5,8} = S(n,5)$

$$\tau = \{\varnothing, A, B, C, D, E, T\}; \qquad \sigma_5 = \frac{9}{4}S(n,3) + 2S(n,4) + S(n,5).$$

Table 2. 5-class

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