Symmetry Lie Group
of the Monge-Ampère Equation

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Abstract

We find the symmetry group of the Monge-Ampère equation and we describe the geometric dynamics induced by an affine vector field.

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A symmetry group of a second order PDE is a group which transforms solutions of the equation to other solutions. We can determine explicitly special types of solutions which are themselves invariant under some subgroup of the full symmetry group. These "group-invariant" solutions are found by solving a reduced system of partial differential equations. The theory of symmetry group for differential equations, partial differential equations and systems has manny applications to geometry, physics and mechanics.

In this paper we shall study the Monge-Ampère equation applying the general theory in the P.J.Olver book[6]. Also we present the geometric dynamics induced by an affine vector field on $\mathbf{R}^{n+1}$. Ours results are original modulo the quoted papers.

The Monge-Ampère equation of two variables is closely related to the Minkovski problem and the Weyl embedding problem. For higher dimension, the Monge-Ampère equation is studied in connection to affine geometry because it is invariant under the special linear group. In this paper, the special linear group is a subgroup of the symmetry group of Monge-Ampère equation.

1 Symmetry Lie group of a PDE of order two

Let $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$, $\pi(x,u) = x$, $x = (x^1, ..., x^n)$, be the projection map.

Let $U \subset \mathbf{R}^{n+1}$ be an open set and $U_0 = \pi(U)$.

Definition 1. A smooth map $s : U_0 \to U$, $s(x) = (x, u(x))$ is called *local section of π (over U_0).*

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For the function u, we note

$$u_{i_1 \dots i_p}(x) = \frac{\partial^p u}{\partial x^{i_1} \dots \partial x^{i_p}}(x),\ x \in U_0,\ 1 \le i_1 \le \dots \le i_p \le n,\ p \ge 1.$$

Let us consider

$$J^k(U) = \{(x^i, u, u_{i_1}, \dots, u_{i_1 \dots i_k}) | (x^i, u) \in U\},$$

$J^k(U) \subset \mathbf{R}^{n+1} \times \mathbf{R}^{N_1} \times \dots \times \mathbf{R}^{N_k}$, where $N_k = \frac{(n+k-1)!}{k!(n-1)!}$, $k > 0$, and the projection

$$\pi^k : J^k(U) \to U_0,\ \pi^k(x, u, u_{i_1}, \dots, u_{i_1 \dots i_k}) = x.$$

Convention: for $k = 0$, $J^0(U) = U$ and $\pi^0 = \pi$.

Definition 2. Let $s : U_0 \to U$ be a local section of π (over U_0). The section $j^k(s)$ of π^k over U_0, called *the k-jet of s*, is defined as follows:

$$j^k(s)(x) = (x, u, u_{i_1}, \dots, u_{i_1 \dots i_k}),\ x \in U_0,$$

$$u_{i_1 \dots i_r} = \frac{\partial^r u}{\partial x^{i_1} \dots \partial x^{i_r}}(p),\ 0 \le r \le k.$$

The space $J^k(U)$ is called *the kth-order jet space.*

Let $\Omega^q_k(U)$ be the vector space of q-forms on $J^k(U)$ with exterior differential d. In particular $\Omega^q_0(U) = \Omega^q(U)$ is the exterior algebra of q-forms on U and $\Omega^0_k = C^\infty(J^k(U))$ is the algebra of real-valued functions

$$f = f(x^i, u, u_{i_1}, \dots, u_{i_1 \dots i_k})$$

on $J^k(U)$. A basis for $\Omega^1_k(U)$ (as a module over $C^\infty(J^k(U))$) consists of the one forms $dx^i,\ du,\ du_{i_1}, \dots, du_{i_1 \dots i_k}$. For $f \in C^\infty(J^k(U))$ we have

$$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial u_{i_1}} du_{i_1} + \dots + \sum_{1 \le i_1 \le \dots \le i_k \le n} \frac{\partial f}{\partial u_{i_1 \dots i_k}} du_{i_1 \dots i_k}.$$

Definition 3. A form $\omega \in \Omega^q_k(U)$ is called *basic* if it is of the type

$$\omega = A_{j_1 \dots j_q} dx^{j_1} \wedge \dots \wedge dx^{j_q},$$

where $A_{j_1 \dots j_q}$ are smooth functions on $J^k(U)$.

Denote the space of basic q-forms by $\mathcal{B}^q_k(U)$.

Definition 4. The operator $D : \mathcal{B}^q_k(U) \to \mathcal{B}^{q+1}_{k+1}(U)$,

$$Df = \left(\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial u} u_j + \frac{\partial f}{\partial u_{i_1}} u_{i_1 j} + \dots + \sum_{1 \le i_1 \le \dots \le i_k \le n} \frac{\partial f}{\partial u_{i_1 \dots i_k}} u_{i_1 \dots i_k j} \right) dx^j,$$

for $f \in C^\infty(J^k(U))$, and

$$D\omega = DA_{j_1 \dots j_q} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q},$$

for

$$\omega = A_{j_1 \ldots j_q} dx^{j_1} \wedge \ldots \wedge dx^{j_q},$$

is called *the total exterior derivative on* $\mathcal{B}_k^q(U)$.

Convention: $Df = D_i f\ dx^i$, where $D_i f$ is the total derivative of f with respect to x^i.

Let $\Omega_2^{n+1}(U)$ be the space of $(n+1)$-forms, and $\theta \in \Omega_2^{n+1}(U)$,

$$\theta = F(x^i, u, u_l, u_{lk}) du \wedge dx^1 \wedge \ldots \wedge dx^n.$$

Definition 5. A solution of the equation

$$\theta = 0$$

on $U_0 \subset \mathbf{R}^n$, is a section $s : U_0 \to U$ such that $\theta \cdot j^2(s) = 0$.

In other words, θ determines the PDE

$$F(x, u, u_l, u_{lk}) = 0, \tag{1}$$

a solution of which is a function $u = u(x)$ such that

$$F\left(x, u(x), \frac{\partial u}{\partial x^l}(x), \frac{\partial^2 u}{\partial x^l \partial x^k}(x)\right) = 0, \forall x \in U_0.$$

We note $u^{(2)} = (u, u_l, u_{lk})$.

Definition 6. The PDE (1) is called of *maximal rank* if the Jacobian matrix

$$J_F(x, u^{(2)}) = (F_{x^i}; F_u; F_{u_i}, F_{u_{ij}})$$

has rank 1 whenever $F(x, u^{(2)}) = 0$.

Then the subset

$$S = \{(x, u^{(2)}) \in J^{(2)}(U) | F(x, u^{(2)}) = 0\}$$

is a hypersurface.

Definition 7. A *symmetry group* of PDE is a local group of transformations G acting on an open set U of the associated space of independent and dependent variables, with the property that whenever $u = f(x)$ is a solution of the equation and whenever $g \cdot f$ is defined for $g \in G$, then $v = g \cdot f(x)$ is also a solution of the equation.

The computational procedure for finding the symmetry group uses the following *infinitesimal criterion of invariance.*

Theorem 1. *Let $F(x, u^{(2)}) = 0$ be a PDE of maximal rank defined on an open set U_0. If G is a local group of transformations acting on U and*

$$pr^{(2)} X[F(x, u^{(2)})] = 0 \quad \textit{whenever} \quad F(x, u^{(2)}) = 0, \tag{2}$$

for every infinitesimal generator X of G, then G is a symmetry group of the equation.

Consider the vector field

$$X = \zeta^i(x, u) \frac{\partial}{\partial x^i} + \phi(x, u) \frac{\partial}{\partial u}$$

on U. *The first prolongation of* X is the vector field

$$\mathrm{pr}^{(1)}X = X + \Phi_i \frac{\partial}{\partial u_i},$$

where

$$\Phi_i = D_i(\phi - \zeta^k u_k) + \zeta^k u_{ik} = \phi_{x^i} + u_i \phi_u - u_k \left(\zeta^k_{x^i} + u_i \zeta^k_u\right).$$

The second prolongation of X is the vector field

$$\mathrm{pr}^{(2)}X = pr^{(1)}X + \Phi_{ij}\frac{\partial}{\partial u_{ij}}, \tag{3}$$

where

$$\begin{aligned}\Phi_{ij} &= D_{ij}(\phi - \zeta^k u_k) + \zeta^k u_{ijk} = \\ &= \phi_{x^i x^j} + u_i \phi_{x^j u} + u_j \phi_{x^i u} + u_i u_j \phi_{uu} + u_{ij}\phi_u - \\ &-u_k\left(\zeta^k_{x^i x^j} + u_i \zeta^k_{x^j u} + u_j \zeta^k_{x^i u} + u_i u_j \zeta^k_{uu} + u_{ij}\zeta^k_u\right) - \\ &-u_{ki}\left(\zeta^k_{x^j} + u_j \zeta^k_u\right) - u_{kj}\left(\zeta^k_{x^i} + u_i \zeta^k_u\right).\end{aligned}$$

Proposition 1. *Let be a PDE of the maximal rank defined on U_0. The set of all infinitesimal symmetries of the equation forms a Lie algebra on U. Moreover, if this Lie algebra is finite-dimensional, the symmetry group of the equation is a local Lie group of transformations acting on U.*

Algorithm for finding the symmetry group of PDE (1)

-one consider a vector field X on U and one writes the infinitesimal invariance condition (2);

-one eliminates any dependence between partial derivatives of the function u, determined by the PDE (1);

-one writes the condition (2) like a polynomial in the partial derivatives of u;

-one equates with zero the coefficients of partial derivatives of u in (2), written as a polynomial in the derivatives of the function u; it follows a PDE system with respect to the unknown functions ζ^i, ϕ, and this system defines the Lie symmetry group G of the given PDE.

Every s-parametric subgroup H of the group G determines a family of group-invariant solutions. The problem of classification of group-invariant solutions reverts to the problem of classification of Lie subalgebras of Lie algebra $\mathbf{g}$ of the group G ([6],pp.186). For 1-dimensional algebras one considers a general element X, and we simplify this as much as possible using the adjoint transformations.

Remark. We will compute the adjoint representation $Ad\ G$ of the underlying Lie group G, by using the Lie series

$$Ad(exp(\varepsilon X)Y) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adX)^n(Y) = Y - \varepsilon[X,Y] + \frac{\varepsilon^2}{2}[X,[X,Y]] - ... \tag{4}$$

2 Symmetry Lie group of the Monge-Ampère equation

The general form of the Monge-Ampère equation is [16]

$$det(u_{ij} + v_{ij}) = H(x, u, u_i),$$

where v_{ij} are functions of x, u, u_i. This is a particular nonlinear second order PDE.

First, let us consider the special case

$$det(u_{ij}) = f(x, u), \tag{5}$$

where U_0 is an open set of $\mathbf{R}^2$, $u \in C^2(U_0)$ and f is a nonconstant function.

In the case of the equation (5) we have

$$F(x, u^{(2)}) = det(u_{ij}) - f(x, u)$$

and the Jacobian matrix is

$$J_F(x, u^{(2)}) = (-f_{x^i}; -f_u; 0; d_{u_{ij}}),\ d = det(u_{ij}).$$

Let A^{ij} be the cofactors of the elements of the determinant d. Because (u_{ij}) is a symmetric matrix, it follows

$$\frac{\partial d}{\partial u_{ij}} = \begin{cases} 2A^{ij} & \text{for} \quad i \neq j \\ A^{ii} & \text{for} \quad i = j. \end{cases}$$

Then for $F(x, u^{(2)}) = 0$, $rank J_F = 1$.

Let X be the vector field

$$X = \zeta^i(x, u)\frac{\partial}{\partial x^i} + \phi(x, u)\frac{\partial}{\partial u}$$

on U. In this case the condition (2) turns in

$$-\sum_{i=1}^{n} \zeta^i f_{x^i} - \phi f_u + \sum_{i=1}^{n} \Phi_{ii} A^{ii} + 2 \sum_{1 \leq i < j \leq n} \Phi_{ij} A^{ij} = 0.$$

Substituing the functions Φ_i and Φ_{ij} defined by the relation (3) and after eliminating any dependencies among the derivatives of the u's caused by the equation (5) itself, we find

$$-\sum_{i=1}^{n} \zeta^i f_{x^i} - \phi f_u + +nf\left(\phi_u - \frac{2}{n}\sum_{i=1}^{n} \zeta^i_{x^i}\right) +$$

$$\sum_{i=1}^{n} A^{ii}\phi_{x^i x^i} + 2 \sum_{i,j=1, i<j}^{n} A^{ij}\phi_{x^i x^j} +$$

$$-(n+2)f\sum_{i=1}^{n} u_i \zeta^i_u + \sum_{i=1}^{n} A^{ii} u_i \left(2\phi_{x^i u} - \zeta^i_{x^i x^i}\right) -$$

$$-\sum_{i,k=1,k\neq i}^{n} A^{ii}u_k\zeta^k_{x^ix^i}+2\sum_{i,j=1,i<j}^{n} A^{ij}u_j\left(\phi_{x^iu}-\zeta^j_{x^ix^j}\right)+$$

$$+2\sum_{i,j=1,i<j}^{n} A^{ij}u_i\left(\phi_{x^ju}-\zeta^i_{x^ix^j}\right)-2\sum_{i,j,k=1,i<j,k\neq i,j}^{n} A^{ij}u_k\zeta^k_{x^ix^j}+$$

$$+\sum_{i=1}^{n} A^{ii}u_i^2\left(\phi_{uu}-2\zeta^i_{x^iu}\right)-2\sum_{i,j=1,i\neq j}^{n} A^{ii}u_iu_j\zeta^j_{x^iu}+$$

$$+2\sum_{i,j=1,i<j}^{n} A^{ij}u_iu_j\left(\phi_{uu}-\zeta^i_{x^iu}-\zeta^j_{x^ju}\right)-$$

$$-2\sum_{i,j=1,i<j}^{n} A^{ij}u_j^2\zeta^j_{x^iu}-2\sum_{i,j=1,i<j}^{n} A^{ij}u_i^2\zeta^i_{x^ju}-$$

$$-2\sum_{i,j,k=1,i<j,k\neq i,j}^{n} A^{ij}u_ku_i\zeta^k_{x^ju}-\sum_{i=1}^{n} A^{ii}u_i^3\zeta^i_{uu}-$$

$$-\sum_{i,k=1,k\neq i}^{n} A^{ii}u_i^2u_k\zeta^k_{uu}-2\sum_{i,j=1,i<j}^{n} A^{ij}u_i^2u_j\zeta^i_{uu}-$$

$$-2\sum_{i,j=1,i<j}^{n} A^{ij}u_iu_j^2\zeta^j_{uu}-2\sum_{i,j,k=1,i<j,k\neq i,j}^{n} A^{ij}u_iu_ju_k\zeta^k_{uu}=0.$$

Now we can equate the coefficients of the remaining unconstrained partial derivatives of u to zero. We obtain a large number of partial differential equations for the coefficient functions ζ^i and ϕ of the infinitesimal operator, namely

$$-\sum_{i=1}^{n}\zeta^i f_{x^i}-\phi f_u+nf\left(\phi_u-\frac{2}{n}\sum_{i=1}^{n}\zeta^i_{x^i}\right)=0,$$

$$\phi_{x^ix^i}=0, i=1,...,n$$

$$\phi_{x^ix^j}=0,\ 1\le i<j\le n,$$

$$\zeta^i_u=0,\ i=1,...,n,$$

$$\phi_{x^iu}-\zeta^j_{x^ix^j}=0,\ 1\le i<j\le n,$$

$$\phi_{x^ju}-\zeta^i_{x^ix^j}=0,\ 1\le i<j\le n,$$

$$\zeta^k_{x^ix^j}=0,\ i,j,k=1,...,n,\ i<j,\ k\neq i,j,$$

$$\phi_{uu}-2\zeta^i_{x^iu}=0,\ i=1,...,n,$$

$$\zeta^j_{x^iu}=0,\ 1\le i<j\le n,$$

$$\phi_{uu}-\zeta^i_{x^iu}-\zeta^j_{x^ju}=0,\ 1\le i<j\le n,$$

$$\zeta^i_{x^ju}=0, 1\le i<j\le n,$$

$$\zeta^k_{x^j u} = 0, j, k = 1, ..., n,\ k \neq j.$$

called *the defining equations* for the symmetry group of the given equation. This PDEs system admits the solutions

$$\zeta^i = x^i \sum_{j=1}^{n} a_j x^j + \sum_{j=1}^{n} b^i_j x^j + c^i,\ \ i = 1, ..., n$$

$$\phi = u\left(\sum_{j=1}^{n} a_j x^j + b\right) + \sum_{j=1}^{n} c_j x^j + c,$$

where $a_j,\ b^i_j,\ b,\ c_j,\ c^i,\ c \in \mathbf{R}$, with

$$-\sum_{i=1}^{n} \zeta^i f_{x^i} - \phi f_u + nf\left(\phi_u - \frac{2}{n}\sum_{i=1}^{n} \zeta^i_{x^i}\right) = 0.$$

In the particular case

$$det(u_{ij}) = 1, \tag{6}$$

the preceding expressions are reduced to

$$\zeta^i = \sum_{j=1}^{n} b^i_j x^j + c^i,\ \ i = 1, ..., n$$

$$\phi = \frac{2u}{n}\sum_{j=1}^{n} b^j_j + \sum_{j=1}^{n} c_j x^j + c,$$

and it follows

$$X = \sum_{i,j=1}^{n} b^j_i\left(x^i \frac{\partial}{\partial x^j} + \delta^i_j \frac{2u}{n}\frac{\partial}{\partial u}\right) +$$

$$+\sum_{i=1}^{n} c_i x^i \frac{\partial}{\partial u} + \sum_{i=1}^{n} c^i \frac{\partial}{\partial x^i} + c\frac{\partial}{\partial u}.$$

Theorem 2. *The Lie algebra of infinitesimal symmetries of the Monge-Ampère equation (6) is spanned by the vector fields*

$$X_{ij} = x^i \frac{\partial}{\partial x^j} + \delta^i_j \frac{2u}{n}\frac{\partial}{\partial u},\ \ 1 \leq i, j \leq n,$$

$$Y_i = x^i \frac{\partial}{\partial u},\ \ 1 \leq i \leq n, \tag{7}$$

$$Z_i = \frac{\partial}{\partial x^i},\ \ 1 \leq i \leq n,$$

$$V = \frac{\partial}{\partial u}.$$

Let consider the case $n = 2$. The equation (6) turns in

$$u_{xx}u_{yy} - u_{xy}^2 = 1. \tag{8}$$

The Lie algebra of infinitesimal symmetries of the Monge-Ampère equation (8) is spanned by nine vector fields

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial u},$$

$$X_4 = y\frac{\partial}{\partial x},\ X_5 = x\frac{\partial}{\partial y}, \tag{9}$$

$$X_6 = x\frac{\partial}{\partial u},\ X_7 = y\frac{\partial}{\partial u},\ X_8 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},$$

$$X_9 = y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u},$$

where we note $x^1 = x,\ x^2 = y$, and $Z_1 = X_1,\ Z_2 = X_2,\ V = X_3,\ X_{21} = X_4,\ X_{12} = X_5,\ Y_1 = X_6,\ Y_2 = X_7,\ X_{11} = X_8,\ X_{22} = X_9$.

Remark. It follows that each one-parameter subgroup G_i generated by X_i is a symmetry group and if $u = f(x, y)$ is a solution of the Monge-Ampère equation (8), then every function

$$u^{(1)} = f(x - \varepsilon, y),\ \ u^{(2)} = f(x, y - \varepsilon),$$

$$u^{(3)} = f(x, y) + \varepsilon,\ \ u^{(4)} = f(x - \varepsilon y, y),$$

$$u^{(5)} = f(x, y - \varepsilon x),\ \ u^{(6)} = f(x, y) + \varepsilon x,$$

$$u^{(7)} = f(x, y) + \varepsilon y,\ \ u^{(8)} = e^{\varepsilon} f(xe^{-\varepsilon}, y),$$

$$u^{(9)} = e^{\varepsilon} f(x, ye^{-\varepsilon}),$$

is another solution for the equation, where ε is a real number.

We compute the adjoint representation *Ad G* of the underlying Lie group G, by using the Lie series (4) and we construct the table

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_1	X_2	X_3	X_4
X_3	X_1	X_2	X_3	X_4
X_4	X_1	$X_2 + \varepsilon X_1$	X_3	X_4
X_5	$X_1 + \varepsilon X_2$	X_2	X_3	$X_4 - \varepsilon(X_8 - \varepsilon X_9) - \varepsilon^2 X_5$
X_6	$X_1 + \varepsilon X_3$	X_2	X_3	$X_4 + \varepsilon X_7$
X_7	X_1	$X_2 + \varepsilon X_3$	X_3	X_4
X_8	$e^{\varepsilon} X_1$	X_2	$e^{\varepsilon} X_3$	$e^{\varepsilon} X_4$
X_9	X_1	$e^{\varepsilon} X_2$	$e^{\varepsilon} X_3$	$e^{-\varepsilon} X_4$

Ad	X_5	X_6	X_7
X_1	$X_5 - \varepsilon X_2$	$X_6 - \varepsilon X_3$	X_7
X_2	X_5	X_6	$X_7 - \varepsilon X_3$
X_3	X_5	X_6	X_7
X_4	$X_5 - \varepsilon(X_9 - X_8) - \varepsilon^2 X_4$	$X_6 - \varepsilon X_7$	X_7
X_5	X_5	X_6	$X_7 - \varepsilon X_6$
X_6	X_5	X_6	X_7
X_7	$X_5 + \varepsilon X_6$	X_6	X_7
X_8	$e^{-\varepsilon} X_5$	X_6	$e^{\varepsilon} X_7$
X_9	$e^{\varepsilon} X_5$	$e^{\varepsilon} X_6$	X_7

Ad	X_8	X_9
X_1	$X_8 - \varepsilon X_1$	X_9
X_2	X_8	$X_9 - \varepsilon X_2$
X_3	$X_8 - \varepsilon X_3$	$X_9 - \varepsilon X_3$
X_4	$X_8 - \varepsilon X_4$	$X_9 + \varepsilon X_4$
X_5	$X_8 + \varepsilon X_5$	$X_9 - \varepsilon X_5$
X_6	X_8	$X_9 - \varepsilon X_6$
X_7	$X_8 - \varepsilon X_7$	X_9
X_8	X_8	X_9
X_9	X_8	X_9

whith the (i,j)-th entry indicating $Ad(exp(\varepsilon X_i))X_j$. For the equation (8) we find several optimal system of one-dimensional subalgebras spanned by

$$X_2 + X_4,\ \ X_8 + X_9,\ \ X_8 - X_9,\ X_3 + X_8 - X_9.$$

In the case of

$$X_2 + X_4 = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

we find the solution

$$u(x,y) = \pm\frac{1}{3}(y^2 - 2x + a)^{-\frac{3}{2}} + b,\ \ a,b \in R,\ \ y^2 - 2x + a > 0.$$

In the case of

$$X_8 + X_9 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u},$$

we get the solution

$$u(x,y) = cx^2 + bxy + \frac{a}{2}y^2,\ \ a,b,c \in R,\ \ a^2 + c^2 > 0,\ 2ac - b^2 = 1.$$

The function u is strictly convex.

Jörgens [4] showed that any convex solution u of the equation (8) defined on $\mathbf{R}^2$ is a convex quadratic polynomial.

In the case of

$$X_8 - X_9 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

we find the following solution

$$u(x,y)=\sqrt{xy(a-xy)}-a\mathrm{arctg}\sqrt{\frac{a-xy}{xy}}+b,\ \ a,b\in R,\ a>0,\ 0<xy<a.$$

For the case of

$$X_3+X_8-X_9=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}+\frac{\partial}{\partial u},$$

we get the next solution $u=u(x,y)$,

$$u=\ln x-\frac{a}{2}\mathrm{arcsin}\left(\frac{2xy+a}{\sqrt{a^2+1}}\right)-\frac{1}{2}\ln\left(1-2axy+\sqrt{1-4axy-4x^2y^2}\right)+$$

$$+\frac{1}{2}\sqrt{1-4axy-4x^2y^2}+b,\ x,y>0,\ \ xy<\frac{-a+\sqrt{a^2+1}}{2},\ \ a>0.\ b\in R.$$

The infinitesimal symmetries of the Monge-Ampère equation (6) can be classified as follows:

1) affine vector fields: $X_{ij}, Y_i.Z_i, V$;

2) Killing vector fields: Z_i, V;

3) solenoidal vector fields: $Z_i, V, Y_i, X_{ij}(i\neq j)$;

4) potential vector fields: Z_i, V, X_{ii};

5) torse forming vector fields: X_{ii} (concurrent), $Y_i, X_{ij}(i\neq j)$ (recurrent), Z_i, V (parallel).

Now we look for the converse of Theorem 2, for the case $n=2$.

Theorem 3. *If the vector fields (9) are infinitesimal symmetries of the PDE*

$$u_{xx}=H(x,y,u,u_x,u_y,u_{xy},u_{yy}),$$

with maximal rank, then this PDE is reduced to the Monge-Ampère equation (8).

Proof. We consider the chain of subalgebras

$$\{X_1\}\subset\{X_1,X_2\}\subset\{X_1,X_2,X_3\}\subset\{X_1,X_5\}\subset\{X_1,X_5,X_6\}$$

$$\subset\{X_1,X_5,X_6,X_8\}\subset\{X_2,X_4,X_5,X_6\},$$

and we impose that the given PDE to be invariant with respect to each subalgebra ([1],pp.303).

1) $\{X_1\}$: $X_1=\dfrac{\partial}{\partial x}$, for which $pr^{(2)}X_1=\dfrac{\partial}{\partial x}$. The condition (2) becomes $pr^{(2)}X_1(F)=0$. It follows

$$F=u_{xx}-H_1(y,u,u_x,u_y,u_{xy},u_{yy}).$$

2) $\{X_1,X_2\}$: $X_2=\dfrac{\partial}{\partial y}$, for which $pr^{(2)}X_2=\dfrac{\partial}{\partial y}$. The condition $pr^{(2)}X_2(F)=0$ implies

$$F=u_{xx}-H_2(u,u_x,u_y,u_{xy},u_{yy}).$$

3) $\{X_1,X_2,X_3\}$: $X_3=\dfrac{\partial}{\partial u}$, for which $pr^{(2)}X_2=\dfrac{\partial}{\partial u}$. The condition (2) is equivalent to $pr^{(2)}X_3(F)=0$. It follows

$$F=u_{xx}-H_3(u,u_x,u_y,u_{xy},u_{yy}).$$

4) $\{X_1, X_5\}$: $X_5 = x\frac{\partial}{\partial y}$, with $pr^{(2)}X_5 = x\frac{\partial}{\partial y} - u_y\frac{\partial}{\partial u_x} - u_{xy}\frac{\partial}{\partial u_{xx}} - u_{yy}\frac{\partial}{\partial u_{xy}}$, for which $pr^{(2)}X_5(F) = 0$ implies

$$F = u_{xx} - \frac{u_{xy}^2}{u_{yy}} - \frac{1}{u_{yy}}H_4(u_y, u_{yy}, u_y u_{xy} - u_x u_{yy}).$$

5) $\{X_1, X_5, X_6\}$: $X_6 = x\frac{\partial}{\partial u}$ and $pr^{(2)}X_6 = x\frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}$ for which $pr^{(2)}X_6(F){=}0$ implies

$$F = u_{xx} - \frac{u_{xy}^2}{u_{yy}} - \frac{1}{u_{yy}}H_5(u_y, u_{yy}).$$

6) $\{X_1, X_5, X_6, X_8\}$: $X_8 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$ and $pr^{(2)}X_8 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x} + u_y\frac{\partial}{\partial u_y} - u_{xx}\frac{\partial}{\partial u_{xx}} + u_{yy}\frac{\partial}{\partial u_{yy}}$. The condition $pr^{(2)}X_8(F) = 0$ implies

$$F = u_{xx} - \frac{u_{xy}^2}{u_{yy}} - \frac{1}{u_{yy}}H_6(\frac{u_{yy}}{u_y}).$$

7) $\{X_2, X_4, X_5, X_6\}$: $X_4 = y\frac{\partial}{\partial x}$ and $pr^{(2)}X_4 = y\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial u_y} - u_{xx}\frac{\partial}{\partial u_{xy}} - 2u_{xy}\frac{\partial}{\partial u_{yy}}$. The condition (2) becomes $pr^{(2)}X_4(F) = 0$. It follows

$$F = u_{xx} - \frac{u_{xy}^2}{u_{yy}} - \frac{k}{u_{yy}}, \quad k \neq 0.$$

Hence

$$u_{xx}u_{yy} - u_{xy}^2 = k.$$

The condition of maximal rank requires $k \neq 0$, and the changing $u \to ku$ shows that we can take $k = 1$.

For $n > 2$, in [9] A.V.Pogorelov showed that a generalized convex solution of the equation (6) is the function

$$u(x) = \left(\sum_{i=2}^{n}(x^i)^2\right)^{1-\frac{1}{n}} \cdot \left(h(x^1)\right)^{\frac{2}{n}-1}, \tag{10}$$

where $h > 0$ is a solution of the ODE $y'' + y^{n-1} = 0$ (particular case of Emden-Fowler equation [5]).The solution of Pogorelov is invariant under the subgrup G_1 generated by

$$X = \sum_{i=2}^{n} X_{ii} = \sum_{i=2}^{n} x^i\frac{\partial}{\partial x^i} + \frac{2(n-1)}{n}u\frac{\partial}{\partial u}.$$

If $g_\varepsilon \in G_1$, $g_\varepsilon(x^1, ..., x^n, u) = (x'^1, ..., x'^n, u')$, $\varepsilon \in \mathbf{R}$, then $x'^1 = x^1 + \varepsilon$, $x'^2 = x^2 e^{\frac{\varepsilon}{2}}$, ..., $x'^n = x^n e^{\frac{\varepsilon}{2}}$, $u' = ue^{\frac{(n-1)\varepsilon}{n}}$. If (10) is a solution it follows that $u' = f(x', y')$ is another solution of the equation. This implies that the solution (10) can be found by using the preceding algorithm for the vector field X.

3 Dynamics induced by an affine vector field

The generic element of the infinitesimal Lie algebra of symmetries of the Monge-Ampère equation is an affine vector field on $\mathbf{R}^{n+1}$.

Let $A = [a^i{}_j]$ be a quadratic matrix of order $n+1$ and $X(x) = Ax + b$ be the associated affine vector field on $\mathbf{R}^{n+1}$. If $f(x) = \frac{1}{2}\|Ax+b\|^2$ is the energy of X, then $D_Y f = (AY, Ax+b)$, $\forall Y \in \mathcal{X}(\mathbf{R}^{n+1})$ and hence zeros of the X (if they exist) are critical points of the energy f. Particularly, the relations

$$D_X X = A(Ax+b), \quad D_X f = (A(Ax), Ax+b)$$

show that the set of the critical points of the energy f contains the field lines of X which are straight lines. The existence of a field line α which is a straight line of $\mathbf{R}^{n+1}$ imposes $rankA \leq n$. If $x_0 \in \mathbf{R}^{n+1}$ is a critical points of the energy f and $rankA = n+1$, then x_0 is a zero of X.

If $X = (X^1, ..., X^{n+1})$, then the gradient of the energy f has the components

$$\frac{\partial f}{\partial x^j} = \sum_{i=1}^{n+1} \frac{\partial X^i}{\partial x^j} X^i = \sum_{i=1}^{n+1} a^i{}_j X^i.$$

The matrix of the Hessian $d^2 f$ of the energy f has the components

$$\frac{\partial^2 f}{\partial x^j \partial x^k} = \sum_{i=1}^{n+1} \frac{\partial X^i}{\partial x^j} \frac{\partial X^i}{\partial x^k} = \sum_{i=1}^{n+1} a^i{}_j a^i{}_k.$$

That in way $d^2 f \geq 0$ and hence the energy f is a convex function on $\mathbf{R}^{n+1}$. This result implies the fact critical points of f coincide with global minimum points of f and hence with zeros of X. Also, the flow determined on $\mathbf{R}^{n+1}$ by $grad\ f$ increases the volume excepting the case in which X is a parallel vector field.

The dynamics induced by a affine vector field is described by the second order differential system in the next [14]

Theorem 4. *Let $X = Ax + b$ be an affine vector field on $\mathbf{R}^{n+1}$ and f be its energy. Any orbit of X is a trajectory of the (potential or nonpotential) dynamical system with $n+1$ degrees of freedom*

$$\frac{d^2 x^i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{n+1} (a^i{}_j - a_j{}^i) \frac{dx^j}{dt}, \quad i = 1, ...n+1, \ where\ a_j{}^i = \delta^{ih} \delta_{kj} a^k{}_h. \tag{11}$$

Theorem 5. *1) The trajectories of the dynamical system (11) are the extremal of the Lagrangian*

$$L = \frac{1}{2}\delta_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} - \delta_{ij} X^i(x) \frac{dx^j}{dt} + f(x).$$

2) The dynamical system (11) is conservative, admitting the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\delta_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} - f(x).$$

Theorem 6 (Lorentz-Udrişte world-force law). *Every nonconstant trajectory of the dynamical system (11), which correspond to a constant value $\mathcal{H}$ of the Hamiltonian, is a reparametrized horizontal geodesic of the Riemann-Jacobi-Lagrange manifold*

$$(R^{n+1} \setminus \mathcal{E}, \quad g_{ij} = (\mathcal{H} + f)\delta_{ij}, \quad N^i{}_j = a^i{}_j - a_j{}^i),$$

where $\mathcal{E}$ is the set of zeros of the vector field X.

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