Computer Simulation of Magnetic Phase Portraits and Geometric Dynamics around Piecewise Rectilinear Circuits

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Abstract

§1 shows that the electric 1-form, the magnetic 1-form, and the Pointing 1-form defines an almost coquaternion metric structure. §2 analyses the electromagnetic dynamics and formulates a Lorenz-Udrişte world-force law. §3 studies the magnetic dynamics around piecewise rectilinear electrical circuits.

Mathematics Subject Classification: 34A50, 58F25, 65L06.
Key Words: Hodge operator, Maxwell equations, almost coquaternion metric structure, electromagnetic dynamics, numerical and graphical simulation.

1 Electromagnetic almost coquaternion metric structure

The volume element defined on R^3 by the Riemannian metric h is defined as $dv = \sqrt{\det h}\ dx^1 \wedge dx^2 \wedge dx^3$. Let η be a p-form and ω be a $(3-p)$-form, $p = 0, 1, 2, 3$. The equation $\eta \wedge \omega = dv$ defines the Hodge duality operator $*$ between p-forms and $(3-p)$-forms. A double transformation of a p-form in a 3-dimensional space restores the original form.

Let $U \subset R^3$ be a domain of linear homogeneous isotropic media. In terms of differential forms, Maxwell equations on $U \times R$ can be expressed as

$$dD = \rho, \quad dH = J + \partial_t D$$
$$dB = 0, \quad dE = -\partial_t B,$$

where B, D, J (respectively, the magnetic induction, electric displacement and electric current density) are all 2-forms; H (the magnetic field) and E (the electric field) are 1-forms; and ρ (the electric charge density) is a 3-form. The operator d is the exterior derivative and the operator ∂_t is the time derivative.

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The constitutive relations are

$$D = \varepsilon {}^*E, \quad B = \mu {}^*H,$$

where the star operator $*$ is the Hodge operator, ε is the permitivity, and μ is the scalar permeability.

Let us introduce the Poynting 2-form $S = E \wedge H$ and its Hodge dual ${}^*S$ which is an 1-form. The 1-forms $\eta_1 = E$, $\eta_2 = H$, $\eta_3 = {}^*S$ are linearly independent at every point of U. Let ξ_a be the dual vector fields, that is,

$$\eta_a(\xi_b) = \delta_{ab}, \quad \sum_a \eta_a \otimes \xi_a = id.$$

We define

$$\phi_a = \xi_c \otimes \eta_b - \xi_b \otimes \eta_c,$$

where $\{a, b, c\}$ is an even permutation of $\{1, 2, 3\}$ and $g = \sum_a \eta_a \otimes \eta_a$. We can verify without difficulty that

$$(\phi_a, \xi_a, \eta_a, g), \quad a = 1, 2, 3$$

is an almost coquaternion metric structure [18] on R^3.

Theorem 1.1 *The electric* 1*-form, the magnetic* 1*-form and the Poynting* 1*-form define an almost coquaternion metric structure on* $U \subset R^3$.

Open problem. Find the physical meaning of the almost coquaternion metric structure in Theorem 1.1.

2 Electromagnetic dynamics

We start with the Riemannian manifold $(U \subset R^3, h)$. The 1-forms E and H are transformed into vector fields via the Riemannian metric h. We denote these vector fields by the same symbols E and H.

Suppose E and H are C^∞ vector fields and denote by Ω the distribution generated by E and H. The generic element in Ω is the vector field

$$X(x,t) = u(x)E(x,t) + v(x)H(x,t), \quad x \in U, \ t \in R.$$

A field line of X is called *electromagnetic line.* More precisely, an electromagnetic line is a solution of the control (kinematic) system

$$\frac{d\gamma}{ds} = u(\gamma(s))E(\gamma(s),t) + v(\gamma(s))H(\gamma(s),t). \tag{1}$$

The set of all electromagnetic lines is called the *electromagnetic phase portrait.*

Let ∇ be the connection induced by the Riemannian metric h. The derivated dynamical system

$$\frac{\nabla}{ds}\frac{d\gamma}{ds} = \frac{\nabla}{ds}(u(\gamma(s))E(\gamma(s),t) + v(\gamma(s))H(\gamma(s),t))$$

represents the dynamics of the kinematic system (1). Introducing the energy $f = \frac{1}{2}h(X,X)$, and the external (1,1)-tensor field $F = \nabla X - h^{-1} \otimes h(\nabla X)$, and using an artifice which does not destroy the prolongation, we can replace the derivated dynamical system by a conservative dynamical system of order 2 fixed by f and F [20].

Theorem 2.1 *The kinematic system* (1) *can be prolonged to (a potential or nonpotential) dynamical system with n degrees of freedom, namely*

$$\frac{\nabla}{ds}\frac{d\gamma}{ds} = grad\, f + F\left(\frac{dx}{dt}\right). \tag{2}$$

We identify the tangent bundle TU with the cotangent bundle T^*U via the Riemannian metric h.

Theorem 2.2 1) *The trajectories of the dynamical system* (2) *are the extremals of the Lagrangian*

$$L = \frac{1}{2}h\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) - g\left(X, \frac{d\gamma}{ds}\right) + f(x).$$

2) *The dynamical system* (2) *is conservative accepting the Hamiltonian*

$$\mathcal{H} = \frac{1}{2}h\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) - f(x).$$

Theorem 2.3 (*Lorentz-Udrişte world-force law*) *Let h_{ij} be the local components of the metric h and Γ^i_{jk}, $i,j,k = 1,2,3$, be the local components of the connection ∇. Every nonconstant trajectory of the dynamical system* (2), *which corresponds to a constant value $\mathcal{H}$ of the Hamiltonian, is a reparametrized horizontal geodesic of the Riemann-Jacobi-Lagrange manifold*

$$(U \setminus \mathcal{E}, \quad \bar{h} = (\mathcal{H} + f)h, \quad N_j{}^i = \Gamma^i_{jk}y^k + F_j{}^i, \quad i,j,k = 1,2,3),$$

where

$$F_j{}^i = \nabla_j X^i - h^{ik}h_{\ell j}\nabla_k X^\ell$$

are the local components of the external tensor field F and $\mathcal{E}$ is the set of zeros of the vector field X.

Particularly, a spatial configuration of piecewise rectilinear electrical circuits Γ produces a magnetic field H on $U = R^3 \setminus \Gamma$ by Biot-Savart-Laplace formula. This field satisfies $\operatorname{curl} H = 0$, $\operatorname{div} H = 0$. The dynamics induced by H is characterized by

$$\begin{aligned}
&\frac{d\gamma}{ds} = H, \quad f = \frac{1}{2}h(H,H), \quad \frac{\nabla}{ds}\frac{d\gamma}{ds} = \operatorname{grad} f,\\
&L = \frac{1}{2}h\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) + f, \quad \mathcal{H} = \frac{1}{2}h\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) - f,\\
&\left(U \setminus \mathcal{E}, \quad \bar{h} = (\mathcal{H} + f)h\right).
\end{aligned}$$

For simplification of computations we accept $h_{ij} = \delta_{ij}$ (the Euclidean metric).

3 Computer simulation of magnetic phase portrait and of magnetic geometric dynamics around a spire of coil

For practical reasons, we consider the magnetic field H of components

$$H_x = -\frac{y}{r_1(r_1 - z)} + \frac{y}{r_2(r_2 - z)} + \frac{z}{r_3(r_3 - y)} - \frac{z}{r_3(r_3 - y + b)} -$$

$$-\frac{z}{r_4(r_4 - y)} + \frac{z}{r_4(r_4 - y + b)}$$

$$H_y = \frac{x + a}{r_1(r_1 - z)} - \frac{x - a}{r_2(r_2 - z)} - \frac{z}{r_3(r_3 - x + a)} + \frac{z}{r_4(r_4 - x - a)}$$

$$H_z = -\frac{x - a}{r_3(r_3 - y)} + \frac{x - a}{r_3(r_3 - y + b)} + \frac{y - b}{r_3(r_3 - x + a)} + \frac{x + a}{r_4(r_4 - y)} -$$

$$-\frac{x + a}{r_4(r_4 - y + b)} - \frac{y - b}{r_4(r_4 - x - a)}$$

defined on $R^3 \setminus \Gamma$, where the configuration Γ is assimilated to a spire of coil (Γ is a four times bended electrical circuit as in Fig. 1).

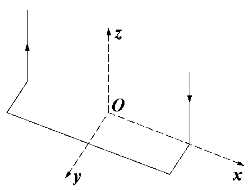


Fig. 1

Figs. 2a - 2d illustrate the magnetic kinematic around coil spire, Fig. 3 the energy diagram, and Fig. 4 the magnetic kinematic determined by a local minimum energy point.

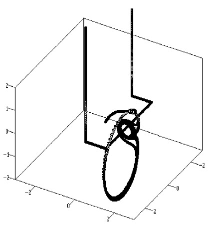


Fig. 2a (frame 30)

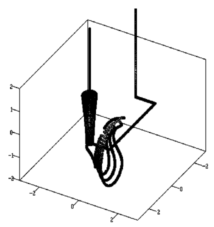


Fig. 2b (frame 72)

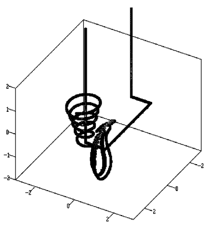


Fig. 2c (frame 105)

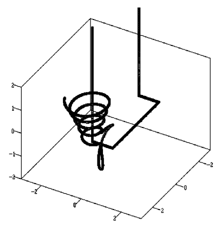
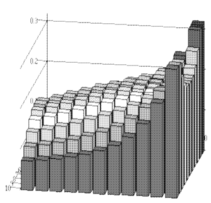


Fig. 2d (frame 120)



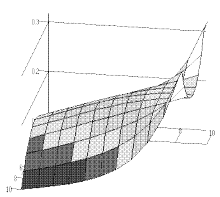


Fig. 3

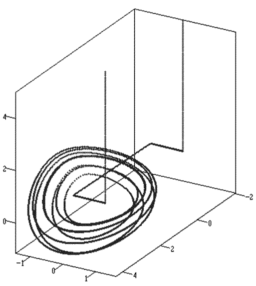


Fig. 4

Since curl $H = 0$, we have

Theorem 3.1 *The nonclassical magnetic dynamics around* Γ *is described by the potential dynamical system with three degrees of freedom*

$$\frac{d^2x}{dt^2} = \frac{\partial f}{\partial x}; \quad \frac{d^2y}{dt^2} = \frac{\partial f}{\partial y}; \quad \frac{d^2z}{dt^2} = \frac{\partial f}{\partial z},$$

where $f = \frac{1}{2}(H_x^2 + H_y^2 + H_z^2)$ *is the energy of the magnetic field* H.

The system in Theorem 3.1 is equivalent to the first order differential system

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w, \quad \frac{du}{dt} = \frac{\partial f}{\partial x}, \quad \frac{dv}{dt} = \frac{\partial f}{\partial y}, \quad \frac{dw}{dt} = \frac{\partial f}{\partial z}.$$

For the computer simulation of solutions of the dynamical system in Theorem 3.1 we need to calculate the partial derivatives of f, and implicitely the derivatives of H. These are

$$\begin{aligned}\frac{\partial H_x}{\partial x} &= \frac{y}{r_1^2(r_1-z)^2}\cdot\frac{x+a}{r_1}(2r_1-z) - \frac{y}{r_2^2(r_2-z)^2}\cdot\frac{x-a}{r_2}(2r_2-z)\\ &- \frac{z}{r_3^2(r_3-y)^2}\cdot\frac{x-a}{r_3}(2r_3-y) + \frac{z}{r_3^2(r_3-y+b)^2}\cdot\frac{x-a}{r_3}(2r_3-y+b)\\ &+ \frac{z}{r_4^2(r_4-y)^2}\cdot\frac{x+a}{r_4}(2r_4-y) - \frac{z}{r_4^2(r_4-y+b)^2}\cdot\frac{x+a}{r_4}(2r_4-y+b);\end{aligned}$$

$$\frac{\partial H_y}{\partial y} = -\frac{x+a}{r_1^2(r_1-z)^2}\cdot\frac{y}{r_1}(2r_1-z) + \frac{z}{r_3^2(r_3-x+a)^2}\cdot\frac{y-b}{r_3}(2r_3-x+a)$$
$$+ \frac{x-a}{r_2^2(r_2-z)^2}\cdot\frac{y}{r_2}(2r_2-z) - \frac{z}{r_4^2(r_4-x-a)^2}\cdot\frac{y-b}{r_4}(2r_4-x-a);$$

$$\frac{\partial H_z}{\partial z} = \frac{x-a}{r_3^2(r_3-y)^2}\cdot\frac{z}{r_3}(2r_3-y) - \frac{x-a}{r_3^2(r_3-y+b)^2}\cdot\frac{z}{r_3}(2r_3-y+b)$$
$$- \frac{y-b}{r_3^2(r_3-x+a)^2}\cdot\frac{z}{r_3}(2r_3-x+a) - \frac{x+a}{r_4^2(r_4-y)^2}\cdot\frac{z}{r_4}(2r_4-y)$$
$$+ \frac{x+a}{r_4^2(r_4-y+b)^2}\cdot\frac{z}{r_4}(2r_4-y+b) + \frac{y-b}{r_4^2(r_4-x-a)^2}\cdot\frac{z}{r_4}(2r_4-x-a);$$

$$\frac{\partial H_y}{\partial x} = \frac{r_1(r_1-z) - \dfrac{(x+a)^2}{r_1}(2r_1-z)}{r_1^2(r_1-z)^2} + \frac{z}{r_3^2(r_3-x+a)^2}\cdot\frac{x-a}{r_3}(2r_3-x+a)$$
$$- \frac{r_2(r_2-z) - \dfrac{(x-a)^2}{r_2}(2r_2-z)}{r_2^2(r_2-z)^2} - \frac{z}{r_4^2(r_4-x-a)^2}\cdot\frac{x+a}{r_4}(2r_4-x-a);$$

$$\frac{\partial H_y}{\partial z} = \frac{x+a}{r_1^3} - \frac{r_3(r_3-x+a) - \dfrac{z^2}{r_3}(2r_3-x+a)}{r_3^2(r_3-x+a)^2}$$
$$- \frac{x-a}{r_2^3} + \frac{r_4(r_4-x-a) - \dfrac{z^2}{r_4}(2r_4-x-a)}{r_4^2(r_4-x-a)^2};$$

$$\frac{\partial H_x}{\partial z} = -\frac{y}{r_1^3} - \frac{r_3(r_3-y+b) - \dfrac{z^2}{r_3}(2r_3-y+b)}{r_3^2(r_3-y+b)^2}$$
$$+ \frac{y}{r_2^3} + \frac{r_4(r_4-y+b) - \dfrac{z^2}{r_4}(2r_4-y+b)}{r_4^2(r_4-y+b)^2}$$
$$+ \frac{r_3(r_3-y) - \dfrac{z^2}{r_3}(2r_3-y)}{r_3^2(r_3-y)^2} - \frac{r_4(r_4-y) - \dfrac{z^2}{r_4}(2r_4-y)}{r_4^2(r_4-y)^2},$$

and consequently

$$\frac{\partial f}{\partial x} = H_x\frac{\partial H_x}{\partial x} + H_y\frac{\partial H_y}{\partial x} + H_z\frac{\partial H_z}{\partial x}$$
$$\frac{\partial f}{\partial y} = H_x\frac{\partial H_x}{\partial y} + H_y\frac{\partial H_y}{\partial y} + H_z\frac{\partial H_z}{\partial y}$$
$$\frac{\partial f}{\partial z} = H_x\frac{\partial H_x}{\partial z} + H_y\frac{\partial H_y}{\partial z} + H_z\frac{\partial H_z}{\partial z}.$$

The computer simulation for the dynamics around coil is now in preparation. To simulate the geometric dynamics we are advanced in establishing a new laboratory endowed with special powerful computer network. For the moment being and to check our ideas, we have used the field generated around a linear wire. The procedure requires the following functions

$$X_1(x,y,z,u,v,w) = u; \quad X_2(x,y,z,u,v,w) = v; \quad X_3(x,y,z,u,v,w) = w;$$
$$X_4(x,y,z,u,v,w) = -2\frac{xy^2}{(x^2+y^2)^3} + \frac{x}{(x^2+y^2)^2} - 2\frac{x^3}{(x^2+y^2)^3};$$
$$X_5(x,y,z,u,v,w) = \frac{y}{(x^2+y^2)^2} - 2\frac{y^3}{(x^2+y^2)^3} - 2\frac{x^2y}{(x^2+y^2)^3};$$
$$X_6(x,y,z,u,v,w) = 0.$$

To carry out our experiments, we have used the iteration as a natural process for computers. To display the images found in this text, we developed software based on adaptive Runge-Kutta type methods. Of course, these programs are by no means optimal in terms of both runtime and usability. Rather, they should serve mainly as guidelines for the reader who wishes to develop software for experimentation purposes.

Breefly, our algorithm performs the following main steps:

STEP 1. Given the second order system in Theorem 3.1 and the matrix of initial conditions which corresponds to the number of particles whose dynamics is studied. Find the matrix of solutions and their projections in three dimensional space.
STEP 2. Set the interval and the integration step.
STEP 3. Apply Runge-Kutta adaptive method and plot.

The dynamics illustrated in Fig. 5 is obtained by using the following three initial conditions:

0	0.5	0	2	0.1	0.1
0	0.5	1	2	0.1	−0.1
−0.544	−0.839	0.5	−0.839	0.544	0

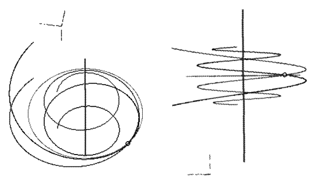


Fig. 5

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