EXTREMA CONSTRAINED BY A FAMILY OF CURVES

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Abstract

§1 analyses extrema constrained by a curve or by a family of curves. §2 introduces and studies the C^1 and C^2 curves containing a given sequence of points. These curves are used to discuss the connection between free extrema and extrema constrained by a family of curves. §3 defines and examines the extrema constrained by a Pfaff inequality.

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1 Introduction

Let us consider the C^1 functions $f, g_1, \ldots, g_q : D \to \mathbf{R}$ on the open set $D \subseteq \mathbf{R}^p$, $q < p$. It is well known that the solution x_0 of the constrained optimum problem *min (max) f subject to* $g_1(x) = \ldots = g_q(x) = 0$, via the method of Lagrange multipliers, leads us to the real numbers $\lambda_1, \ldots, \lambda_q$ with the property

$$df(x_0) + \lambda_1 dg_1(x_0) + \ldots + \lambda_q dg_q(x_0) = 0.$$

Since the submanifold of $\mathbf{R}^p$ defined by the equations $g_1(x) = 0, \ldots, g_q(x) = 0$ is an integral manifold of the Pfaff system $dg_1 = 0, \ldots, dg_q = 0$, we can introduce the notion of *extremum problem constrained by a Pfaff system.*

Let

$$(1) \qquad \omega^j = \sum_{i=1}^{p} \omega_i^j(x) dx^i = 0 \quad j = \overline{1,q}, \quad q < p$$

be a Pfaff system, where $\omega_i^j : D \to \mathbf{R}$ are C^1 functions such that

$$\text{rank}\,[\omega_i^j(x)] = q, \quad \forall x \in D,$$

and let $f : D \to \mathbf{R}$. What is the meaning of the problem

$$min(max) f \text{ } constrained \text{ } by \text{ } \omega^j = 0, \quad j = \overline{1,q},$$

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or

$$min(max) f \text{ constrained by } \omega^j \geq 0, \quad j = \overline{1,q}?$$

Because the solutions of the system (1) are organized in the so called integral manifolds, which in the case of noncompletely integrable system may have the dimension between 1 and $p-q-1$, we start with extrema constrained by a curve or a family of curves, and after that we study extrema constrained by a Pfaff inequality. We recall some well known definitions.

1.1. Definition. A function $\alpha : I \to \mathbf{R}^p$, where I is an open interval and α is a function of suitable class (at least C^0), is called *parametrized curve.*

1.2. Definition. Two parametrized curves $\alpha : I \to \mathbf{R}^p$ and $\beta : J \to \mathbf{R}^p$ are *equivalent* if there exists a one-to-one mapping $\varphi : I \to J$ of the same class with α and β such that $\alpha = \beta \circ \varphi$. We denote $\alpha \sim \beta$, and we have an *equivalence relation.*

1.3. Definition. An equivalence class $\tilde{\alpha}$ of a given parametrized C^k curve α is called *curve.* Then α is called a representative of $\tilde{\alpha}$.

Because any C^0 one-to-one mapping $\varphi : I \to J$ is monotone we have

1.4. Definition. Two equivalent parametrized curves α and β have the *same orientation* if the mapping φ is strictly increasing. If the mapping φ is a strictly decreasing one says that α and β have *oposite orientation.*

1.5. Definition. An equivalence class of C^k parametrized curves having the same orientation is called *oriented C^k curve.*

Every parametrized curve α defines two oriented curves $\tilde{\alpha}_+ = \{\beta \in \tilde{\beta} \mid \beta$ has the same orientation as $\alpha\}$ and $\tilde{\alpha}_- = \{\beta \in \tilde{\alpha} \mid \beta$ and α have oposite orientation$\}$, which are called, respectively, *positive* and *negative* orientation of the parametrized curve α.

We say that the parametrized curve α *is passing through the point* $x_0 \in \mathbf{R}^p$ if there exists $t_0 \in I$ such that $\alpha(t_0) = x_0$. We say that a curve $\tilde{\alpha}$ (oriented curve $\tilde{\alpha}_+$) is passing through the point $x_0 \in \mathbf{R}^p$ if the representative α is passing through x_0.

Let $f : D \subseteq \mathbf{R}^p \to \mathbf{R}$ be a function on the open set D, $x_0 \in D$ and $\alpha : I \to \mathbf{R}^p$ a parametrized curve passing through $x_0 = \alpha(t_0), \quad t_0 \in I$.

1.6. Definitions. a) The point x_0 is called a *minimum (maximum) point for f constrained by the parametrized curve* α, if there exists $\varepsilon > 0$ such that $f(\alpha(t)) \geq f(x_0)$ $(f(\alpha(t)) \leq f(x_0))$ for each $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \subseteq I$.

b) The point x_0 is called a *minimum (maximum) point for f constrained by the oriented curve* $\tilde{\alpha}_+$, if there exists $\varepsilon > 0$ such that $f(\alpha(t)) \geq f(x_0)$ $(f(\alpha(t)) \leq f(x_0))$ for each $t \in [t_0, t_0 + \varepsilon) \subseteq \dot{I}$.

c) The point x_0 is called a *minimum (maximum) point for f constrained by the curve* $\tilde{\alpha}$, if x_0 is a minimum (maximum) point for the function f resticted to the set $\alpha(\dot{I})$.

Observe that, in the case a), if x_0 is an extremum point for f constrained by the parametrized curve α, then x_0 is an extremum point for f constrained by any parametrized curve which is in $\tilde{\alpha}$. Also, in the case b), the definition does not depend upon the element $\beta \in \tilde{\alpha}_+$.

If we denote Γ_{x_0} a family of parametrized curves (oriented curves, curves) passing through the point x_0, then we accept

1.7. Definition. The point x_0 is called a *minimum (maximum) point for the function f constrained by the family* Γ_{x_0}, if x_0 is a minimum (maximum) point for f constrained by each element of the family Γ_{x_0}.

1.8. Proposition. Let $f : D \subseteq \mathbf{R}^p \to \mathbf{R}$ be a function on the open set D, $x_0 \in D$ and α a parametrized curve passing through x_0.

The following properties are equivalent:

1) x_0 is a minimum (maximum) point for f constrained by the parametrized curve α.

2) x_0 is a minimum (maximum) point for f constrained by the oriented curves $\tilde{\alpha}_+$ and $\tilde{\alpha}_-$.

The proof is obvious.

2 C^1 and C^2 curves passing through a given sequence of points

The aim of this paragraph is to prove that some conditions, enough less restrictive, ensure the existence of a parametrized C^1 or C^2 curve which is passing through a given sequence of points. The existence of such a curve is needed to prove the connection between the extremum points constrained by a family of parametrized curves and the free extremum points.

In [15] we proved

2.1 Theorem. *Let (x_n), (y_n) be two sequences of non-zero, real numbers such that* $\lim x_n = \lim y_n = \lim \dfrac{y_n}{x_n} = 0$. *Then*

a) there exist the subsequences (x_{n_k}), (y_{n_k}) and a function $f : \boldsymbol{R} \to \boldsymbol{R}$ of the class C^1 such that

$$f(x_{n_k}) = y_{n_k}, \quad \forall k \in \boldsymbol{N},$$

$$f(0) = f'(0) = 0;$$

b) if, moreover, $x_n > 0$, $y_n > 0$, $\forall n \in \boldsymbol{N}$, then the above function f can be chosen a non-decreasing one;

c) there exist the subsequences (x_{n_k}), (y_{n_k}), the C^2 functions $f, g : \boldsymbol{R} \to \boldsymbol{R}$, and a sequence (t_k) of positive real numbers, with $\lim t_k = 0$, such that

$$f(t_k) = x_k, \quad g(t_k) = y_k,$$

$$f'(0) = g'(0) = g''(0) = 0 \quad \text{and} \quad f''(0) \neq 0.$$

Based, on the above theorem, we obtain

2.2. Theorem. *Let (x_n) be a sequence of distinct points in $\boldsymbol{R}^p$ with $\lim x_n = a$. Then*

a) there exist a subsequence (x_{n_k}) and a parametrized C^1 curve $\alpha : \dot{I} \subseteq \boldsymbol{R} \to \boldsymbol{R}^p$ passing through the points x_{n_k} and a, $\forall k \in \boldsymbol{N}$, which is regular at the point a (i.e., if $a = \alpha(t_0)$, then there exists the sequence $(t_k) \subseteq \dot{I}$ such that $\lim t_k = t_0$, and $\alpha(t_k) = x_{n_k}$ and $\alpha'(t_0) \neq 0$).

b) there exist a subsequence (x_{n_k}) and a parametrized C^2 curve $\alpha : \dot{I} \subseteq \boldsymbol{R} \to \boldsymbol{R}$ passing through the points a and x_{n_k}, $\forall k \in \boldsymbol{N}$, which has a tangent at the point a (i.e., $\alpha'(t_0) = 0$, $\alpha''(t_0) \neq 0$).

Proof. a) By a translation, we can suppose $a = (0, \ldots, 0) \in \mathbf{R}^p$. Because the sequence $u_n = x_n/\|x_n\|$ is bounded, we can consider $u_n \to u \in \mathbf{R}^p$ and by a rotation we have $u = (1, 0, \ldots, 0)$. Then, if $x_n = (x_n^1, \ldots, x_n^p)$, it results

$$\lim \frac{x_n^1}{|x_n^1|\sqrt{1 + \left(\frac{x_n^2}{x_n^1}\right)^2 + \ldots + \left(\frac{x_n^p}{x_n^2}\right)^2}} = 1,$$

and hence $x_n^1 > 0$, for n sufficient large, and $x_n^i/x_n^1 \to 0$, $i = \overline{1,p}$. Applying the above theorem for the sequences (x_n^1), (x_n^i), $i = \overline{2,p}$, we obtain the subsequence $x_{n_k} = \left(x_{n_k}^1, \ldots, x_{n_k}^p\right)$ and the C^1 functions $\varphi_i : \mathbf{R} \to \mathbf{R}$ such that

$$\varphi_i(x_{n_k}^1) = x_{n_k}^i, \quad \forall k \in \mathbf{N}, \quad i = \overline{2,p},$$

$$\varphi_i(0) = \varphi_i'(0) = 0, \quad i = \overline{2,p}.$$

The required parametrized curve will be given by

$$\alpha(t) = (t, \varphi_2(t), \ldots, \varphi_p(t)).$$

b) As in the first part of this proof we apply the theorem 2.1 b) for the sequences $(x_n^1), (x_n^i)$, $i = \overline{2,n}$, and we shall obtain the C^2 functions $\varphi_i : \mathbf{R} \to \mathbf{R}$, $i = \overline{1,p}$, and the sequence $(t_k) \subseteq \mathbf{R}$, $t_k \to 0$, such that

$$\varphi_i(t_k) = x_{n_k}^i,$$

$$\varphi_i(0) = \varphi_i'(0) = 0, \quad i = \overline{1,p},$$

$$\varphi_1''(0) = \frac{1}{2}, \quad \varphi_i''(0) = 0, \quad i = \overline{2,p}.$$

Then the parametrized curve $\alpha(t) = (\varphi_1(t), \ldots, \varphi_p(t))$ has the required properties.

2.3. Theorem. *Let $f : D \subseteq \boldsymbol{R}^p \to \boldsymbol{R}$ be a function on the open set D and $x_0 \in D$. Then the following properties are equivalent:*

1) x_0 is a free local minimum (maximum) point for f.

2) x_0 is a minimum (maximum) point for f constrained by the family of all parametrized C^1 curves passing through x_0 and which are regular at this point.

3) x_0 is a minimum (maximum) point for f constrained by the family of all parametrized C^2 curves, passing through x_0, which have tangents at this point.

Proof. 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 3) are obvious.

2) $\Rightarrow$ 1). We can assume $f(x_0) = 0$. If x_0 would not be a free local minimum point for f, then a sequence (x_n) of distinct points in $\mathbf{R}^p$ would exist for which $\lim x_n = x_0$ and $f(x_n) < 0$. Then the parametrized curve $\alpha : \mathbf{R} \to \mathbf{R}^p$, as in the theorem 2.2 a), would show us that $x_0 = \alpha(0)$ would not be a minimum point for f constrained by α.

3) $\Rightarrow$ 1) follows as the above.

Let us note $(\mathbf{R}_m^p)^+ = \{x = (x^1, \ldots, x^p) \in \mathbf{R}^p / x^{m+1} \geq 0, \ldots, x^p \geq 0\}$.

2.4. Theorem. *Let $(x_n) \subseteq (\boldsymbol{R}_m^p)^+$ be a sequence of distinct points with $\lim x_n = 0 \in \boldsymbol{R}^p$. Then there exist a subsequence (x_{n_k}), a parametrized C^1 curve $\alpha : \boldsymbol{R} \to \boldsymbol{R}^p$, and a sequence $(t_k) \subseteq \boldsymbol{R}$, $t_k > 0$, $t_k \to 0$, such that $\alpha(0) = 0$, $\alpha(t_k) = x_{n_k}$ and $\alpha(t) \in (\boldsymbol{R}_m^p)^+$ for any $t \geq 0$ and α is regular at the point 0.*

Proof. We shall prove that if the sequence (x_n) has the property $x_n^p \geq 0$, then the parametrized curve $\alpha(t) = (x^1(t), \ldots, x^p(t))$, obtained as in theorem 2.2, has the property $x^p(t) \geq 0$, $\forall t \geq 0$. Suppose that $x_n^p > 0$, $\forall n \in \mathbf{N}$ (if (x_n) would contain a subsequence (x_{n_k}) with $x_{n_k}^p = 0$, then we put $x^p(t) = 0$.) Let us consider $u_n = x_n/\|x_n\|$ which, being bounded, can be supposed convergent to a versor $u = (u^1, \ldots, u^p)$. We have $u^p \geq 0$.

The case $u^p = 0$. By a rotation in the subspace $x^p = 0$ we can have $u = (1, 0, \ldots, 0)$. Then following the proof of the theorem 2.2 we have $u_n^p \to 0$, and $x_n^1/\|x_n\| \to 1$, hence $x_n^1 > 0$. By the theorem 2.1 b), for the sequences (x_n^1) and (x_n^p), we obtain the function $\varphi_p : \mathbf{R} \to \mathbf{R}$ which is nondecreasing and $\varphi_p(0) = 0$. It follows $\varphi_p(t) \geq 0$, $\forall t \geq 0$, and hence the parametrized curve $\alpha(t) = (t, \varphi_2(t), \ldots, \varphi_p(t))$ as in the theorem 2.2 is the required one.

The case $u^p > 0$. By a rotation we can have $u = (1, 0, \ldots, 0)$. By this rotation the halfspace $x^p > 0$ becomes a halfspace $h(x^1, \ldots, x^p) > 0$, where $h(x^1, \ldots, x^p) = \sum_{i=1}^{p} a_i x^i$. We have, in this case, $a_1 = h(u) > 0$.

The parametrized curve $\alpha(t) = (t, \varphi_2(t), \ldots, \varphi_p(t))$ as in the theorem 2.2 has the property $\alpha'(0) = (1, 0, \ldots, 0) = u$. Let be $\psi(t) = h(\alpha(t))$. Then $\psi'(0) = a_1 > 0$. It results that in a neighborhood of $t = 0$ we have $\psi'(t) > 0$, too. Because the sequence (t_k) is so $t_k \to 0$ and $t_k > 0$ we can suppose $\psi'(t) > 0$ for $t \in (-\varepsilon, t_1)$. If we modify the curve α such that $\alpha'(t) = (1, 0, \ldots, 0)$ for $t \geq t_1$, it results $\psi'(t) > 0$, $\forall t \geq 0$, and so ψ is an increasing function, and hence $h(\alpha(t)) = \psi(t) > 0$, $\forall t > 0$. So α has the required properties.

2.6. Corollary. *Let $D \subseteq \boldsymbol{R}^p$ be an open set, $a \in D$, let $g_i : D \to \boldsymbol{R}$, $i = \overline{m+1, p}$ be C^1 functions with rank$\left[\dfrac{\partial g_i}{\partial x^j}(a)\right] = p - m$ and $(x_n) \subseteq D$ a sequence of different points, $x_n \to a$, with the property $g_i(x_n) \geq 0$, $i = \overline{m+1, p}$, $n \in \boldsymbol{N}$. Then there exist a subsequence (x_{n_k}), a sequence $t_k \to 0$ of positive real numbers and a regular parametrized C^1 curve $\alpha : \boldsymbol{R} \to D$, such that $\alpha(0) = a$, $\alpha(t_k) = x_{n_k}$ and $g_i : (\alpha(t)) \geq 0$, $\forall t \geq 0$, $i = \overline{m+1, p}$.*

Proof. If $g_i(a) > 0$, $i = \overline{m+1, p}$, one can apply directly the theorem 2.2. If for a function g_i we have $g_i(a) = 0$, then we make the change of variables $y = G(x)$,

$$y^i = x^i, \qquad \text{for } i \leq m, \text{ or } i \geq m+1 \text{ and } g_i(a) > 0$$
$$y^i = g_i(x), \qquad \text{for } i \geq m+1, \text{ and } g_i(a) = 0,$$

which is a C^1 diffeomorphism. Now we can apply the above theorem for the sequence $y_n = G(x_n)$.

3 Extrema constrained by a Pfaff inequality

Let $D \subseteq \mathbf{R}^p$ be an open set and the Pfaff forms

$$(S) \qquad \omega^j = \sum_{i=1}^{p} \omega_i^j(x) dx^i, \quad j = \overline{1, q},\ q < p,$$

where $\omega_i^j : D \to \mathbf{R}$ are C^1 functions in D and rank $[\omega_i^j(x)] = q$.

3.1. Definition. If $f : D \to \mathbf{R}$ is a function, we say that the point $x_0 \in D$ is a *minimum point for f constrained by the system of "inequalities"*

$$(S^+) \qquad \omega^j \geq 0, \quad j = \overline{1,q},$$

if for each parametrized C^2 curve, $\alpha : \dot{I} \to D$, which is passing through x_0 ($\alpha(t_0) = x_0$) the condition

$$(+) \qquad \int_{t_0}^{t} < \omega^j(\alpha(\tau)), \quad \alpha'(\tau) > d\tau \geq 0, \quad \forall t \geq t_0,$$

implies $f(\alpha(t)) \geq f(x_0), \quad \forall t \in [t_0, t_0 + \varepsilon)$.

If the inequality $(+)$ is replaced by the oposite one, we obtain the notion of the minimum point constrained by

$$(S^-) \qquad \omega^j \leq 0.$$

3.2. Remark. a) Suppose that the system (S) is a completly integrable one, i.e., there exist $g^j : D \to \mathbf{R}$ of the class C^1, with $\omega^j = dg^j$. If $x_0 \in D$ is a minimum point for f constrained by $g^j(x) - g^j(x_0) \geq 0$, $j = \overline{1,q}$, then x_0 is a minimum point for f constrained by $\omega^j \geq 0$ and, conversely, in the case when we consider all the C^1 curves by theorem 2.3.

b) Clearly, if a parametrized curve $\tilde{\alpha}$ fullfils the condition $(+)$, then the oriented curve $\tilde{\alpha}_+$ is so, but $\tilde{\alpha}_-$ is not. Hence the condition $(+)$ defines a family of oriented curves passing through x_0.

3.3. Definition. The point $x_0 \in D$ is called an *extremum point for the function $f : D \to \mathbf{R}$ constrained by the system (S)*, if x_0 is an extremum point for f constrained by the family of all C^2 integral curves of the system (S) which are passing through x_0.

3.4. Proposition. *If x_0 is an extremum point for f constrained by (S^+), then x_0 is an extremum point for f constrained by (S).*

Proof. Indeed, suppose that x_0 is a minimum point for f constrained by (S^+). Let $\alpha : \dot{I} \to \mathbf{R}$ be C^2 integral curve of the system (S) passing through x_0 ($\alpha(0) = x_0$), which is regular at x_0. The curve α fullfils the condition $(+)$, so that it follows $f(\alpha(t)) \geq f(x_0)$, $\forall t \in [0, \varepsilon_1)$. On the other hand the parametrized curve $\beta(t) = \alpha(-t)$ is an integral curve and it fullfils also the condition $(+)$. Hence $f(\beta(t)) \geq f(x_0)$, $\forall t \in [0, \varepsilon_2)$, so is $f(\alpha(-t)) \geq f(x_0)$, $\forall t \in [0, \varepsilon_2)$ and finally $f(\alpha(t)) \geq f(x_0)$, $\forall t \in (-\varepsilon_2, \varepsilon_1)$.

3.5. Theorem. *Let $f : D \to \mathbf{R}$ be a C^1 function and $x_0 \in D$ an extremum point for f constrained by (S^+). Then it exist $\lambda_1 \geq 0, \ldots, \lambda_q \geq 0$ such that*

$$df(x_0) = \sum_{j=1}^{q} \lambda_j \omega^j(x_0).$$

Proof. Suppose x_0 a minimum point. Let $v \in \mathbf{R}^p$ be such that

$$< \omega^j(x_0), v > \geq 0, \quad j = \overline{1,q},$$

and $J_0 = \{j = \overline{1,q} \,|< \omega^j(x_0), v >= 0\}$.

Let $\alpha : I \to D$ be a C^2 integral curve of the Pfaff system $\omega^j(x) = 0$, $j \in J_0$, with $\alpha(0) = x_0$, $\alpha'(0) = v$ (if $J_0 = \emptyset$, then α is an arbitrary curve of the class C^2 with

$\alpha(0)=x_0$ and $\alpha'(0)=v$). For any $j\notin J_0$ it follows $<\omega^j(x_0),\ \alpha'(0)>>0$ and hence there exists $\eta>0$ such that $<\omega^j(\alpha(t)),\ \alpha'(t)>>0$, $\forall t\in(-\eta,\eta)$ and $\forall j\notin J_0$. Then we have

$$\int_0^t <\omega^j(\alpha(\tau)),\ \alpha'(\tau)>d\tau>0,$$

for $j=\overline{1,q}$ and $t\in[0,\eta)$.

It follows $\varphi(t)=f(\alpha(t))\geq f(x_0)=\varphi(0),\quad \forall t\in[0,\varepsilon)$, where $\varepsilon<\eta$. Hence we have $\varphi'(0)\geq 0$ and finally $<df(x_0),v>\geq 0$. The conclusion results now by Farkas' lemma.

3.6. Lemma. *Let $a_1,\dots,a_q$, c points in the vector space $\boldsymbol{R}^p$ and $m\leq q$ such that for any $x\in\boldsymbol{R}^p$, $<a_j,x>\geq 0$ implies $<c,x>=0$, $j=\overline{1,m}$, and $<a_j,x>=0$ implies $<c,x>=0$, $j=\overline{m+1,q}$. Then there exist the real numbers $\lambda_1,\dots,\lambda_q$, with $\lambda_j\geq 0$ for $j=\overline{1,m}$, such that $c=\sum\limits_{j=1}^{q}\lambda_j a_j$.*

Proof. Let us consider the points in $\mathbf{R}^p$

$$b_j=\begin{cases} a_j, & j=\overline{1,q}\\ -a_{m+j-q}, & j=\overline{q+1,2q-m}.\end{cases}$$

Then, for any $x\in\mathbf{R}^p$, the inequalities $<b_j,x>\geq 0$, $j=\overline{1,2q-m}$, imply $<c,x>\geq 0$. By Farkas' lemma it follows that there exist $\mu_j\geq 0$, $j=\overline{1,2q-m}$, such that

$$c=\sum_{j=1}^{2q-m}\mu_j b_j=\sum_{j=1}^{m}\mu_j b_j+\sum_{j=m+1}^{q}(\mu_j-\mu_{q+j-m})a_j.$$

3.7. Proposition. *Let $f:D\subseteq\boldsymbol{R}^p\to\boldsymbol{R}$ be a C^1 function and $\omega^j:D\to\boldsymbol{R}$, $j=\overline{1,q}$, be Pfaff forms with coefficients of the class C^1 and rank $[\omega_i^j(x)]=q<p$, $\forall x\in D$. If x_0 is an extremum point for f constrained by*

$$\begin{cases}\omega^j\geq 0, & \text{for } j=\overline{1,m}\\ \omega^j=0, & \text{for } j=\overline{m+1,q},\end{cases}$$

$m\leq q$, then there exist the real numbers $\lambda_1,\dots,\lambda_q$, with $\lambda_j\geq 0$ for $j=\overline{1,m}$, such that

$$df(x_0)=\sum_{j=1}^{q}\lambda_j\omega^j(x_0).$$

The proof is as in the theorem 3.5, using the above lemma.

3.8. Proposition. *The point $x_0\in D$ is a minimum point for the function $f:D\to R$ constrained by the system (S) iff x_0 is a minimum point for f constrained by the system of inequalities (S^+) and (S^-).*

Proof. Let x_0 be a minimum point for f constrained by the system $\omega^j(x)=0$, $j=\overline{1,q}$. If $\alpha:I\to D$ is a C^2 regular curve passing through x_0 for which we have simultaneously

$$\int_{t_0}^{t}<\omega^j(\alpha(\tau),\ \alpha'(\tau))d\tau\geq 0,\quad j=\overline{1,q}$$

and

$$\int_{t_0}^{t} < \omega^j(\alpha(\tau)),\ \alpha'(\tau) > d\tau \le 0, \quad j = \overline{1,q},$$

for $t \ge t_0$, there results $< \omega^j(\alpha(t)),\ \alpha'(t) >= 0$, $\forall t \ge t_0$.

Let $\beta : I \to D$ be an integral curve for the system (S), with $\alpha(t_0) = x_0$ and $\beta'(t_0) = \alpha'(t_0)$. Then the curve

$$\gamma(t) = \begin{cases} \beta(t), & t < t_0 \\ \alpha(t), & t \ge t_0 \end{cases}$$

is an integral curve for the system (S), with $\gamma(t_0) = x_0$. By the hypothesis, there results $f(\gamma(t)) \ge f(x_0)$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, so is $f(\alpha(t)) \ge f(x_0)$, $\forall t \in [t_0, t_0 + \varepsilon)$. This means x_0 is a minimum point for f constrained by the system (S^+) and (S^-).

The converse is obvious because for any integral curve, $\alpha : I \to D$, of the system S we have

$$< \omega^j(\alpha(t)),\ \alpha'(t) >= 0, \quad \forall t \in I, \quad j = \overline{1,q}.$$

3.9. Theorem. *Let $f : D \subseteq \boldsymbol{R}^p \to \boldsymbol{R}$ be a C^2 function $x_0 \in D$ and (S) a C^1 Pfaff system in D. If:*

1) there exist the real numbers $\lambda_j \ge 0$, $j = \overline{1,q}$ such that

$$df(x_0) = \sum_{j=1}^{q} \lambda_j \omega^j(x_0),$$

2) the restriction of the quadratic from

$$d^2 f(x_0) - \frac{1}{2} \sum_{j=1}^{q} \lambda_j \sum_{r,s=1}^{p} \left(\frac{\partial \omega_r^j}{\partial x^s} + \frac{\partial \omega_s^j}{\partial x^r} \right) (x_0) dx^r dx^s$$

to the subspace

$$\sum_{i=1}^{p} \omega_i^j(x_0) dx^i = 0, \quad j \in J' = \{ j = \overline{1,q} \mid \lambda_j > 0 \}$$

is positive definite, then x_0 is a minimum point for f constrained by the inequalities $\omega^j \ge 0$, $j = \overline{1,q}$.

Proof. Let $\alpha : I \to D$ be a C^2 curve, with $\alpha(t_0) = x_0$, which is regular at x_0, and satisfying

$$\int_{t_0}^{t} < \omega^j(\alpha(\tau),\ \alpha'(\tau) > d\tau \ge 0, \quad \forall t \ge t_0,\ j = \overline{1,q}.$$

Case 1. If there exists $j_0 \in J'$ such that

$$< \omega^{j_0}(x_0),\ \alpha'(t_0) >> 0,$$

then

$$df(x_0)(\alpha'(t_0)) = \sum_{j=1}^{q} \lambda_j < \omega^{j_0}(x_0),\ \alpha'(t_0) >> 0,$$

Using Taylor expansion

$$f(x) - f(x_0) = df(x_0)(x - x_0) + \mathcal{O}(\|x - x_0\|)$$

and

$$\alpha(t) - \alpha(t_0) = \alpha'(t_0)(t - t_0) + g(t) \cdot (t - t_0),$$

with $\lim_{t \to t_0} g(t) = 0$, there results

$$f(\alpha(t)) - f(\alpha(t_0)) = (t - t_0)df(x_0)(\alpha'(t_0)) + (t - t_0)df(x_0)(g(t)) +$$

$$+\mathcal{O}(\|\alpha(t) - \alpha(t_0)\|) = (t - t_0)df(x_0)(\alpha'(t_0)) + \mathcal{O}(t - t_0) \geq 0, \quad \forall t \in [t_0, t_0 + \varepsilon).$$

Case 2. Suppose

$$< \omega^j(x_0),\ \alpha'(t_0) >= 0, \quad \forall j \in J'.$$

Let us consider the function

$$\varphi(t) = f(\alpha(t)) - \sum_{j=1}^{q} \lambda_j \int_{t_0}^{t} < \omega^j(\alpha(\tau)),\ \alpha'(\tau) > d\tau.$$

Then

$$\varphi'(t) = \sum_{i=1}^{p} \frac{\partial f}{\partial x^i}(\alpha(t)) \frac{dx^i}{dt} - \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{p} \omega_i^j(\alpha(t)) \frac{dx^i}{dt},$$

whence

$$\varphi'(t_0) = \sum_{i=1}^{p} \left(\frac{\partial f}{\partial x^i}(x_0) - \sum_{j=1}^{q} \lambda_j \omega_i^j(x_0) \right) \frac{dx^i}{dt} =$$

$$= (df(x_0) - \sum_{j=1}^{q} \lambda_j \omega^j(x_0))(\alpha'(t_0)) = 0.$$

Also

$$\varphi''(t) = \sum_{r,s=1}^{p} \frac{\partial^2 f}{\partial x^r \partial x^s}(\alpha(t)) \frac{dx^r}{dt} \frac{dx^s}{dt} + \sum_{i=1}^{p} \frac{\partial f}{dx^i}(\alpha(t)) \frac{d^2 x^i}{dt^2} -$$

$$-\frac{1}{2} \sum_{j=1}^{q} \lambda_j \sum_{r,s=1}^{p} \left(\frac{\partial \omega_r^j}{\partial x^s} + \frac{\partial \omega_s^j}{\partial x^r} \right) (\alpha(t)) \frac{dx^r}{dt} \frac{dx^s}{dt} + \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{p} \omega_i^j(\alpha(t)) \frac{d^2 x^i}{dt^2}.$$

Then

$$\varphi''(t_0) = d^2 f(x_0) - \frac{1}{2} \sum_{j=1}^{q} \lambda_j \sum_{r,s=1}^{p} \left(\frac{\partial \omega_r^j}{\partial x^s} + \frac{\partial \omega_s^j}{\partial x^r} \right) (x_0) \frac{dx^r}{dt}(t_0) \cdot \frac{dx^s}{dt}(t_0) +$$

$$+ \sum_{i=1}^{p} \left(\frac{\partial f}{\partial x^i}(x_0) - \sum_{j=1}^{q} \lambda_j \omega_i^j(x_0) \right) \frac{d^2 x^i}{dt^2}(t_0) =$$

$$= d^2 f(x_0) - \frac{1}{2} \sum_{j=1}^{q} \lambda_j \sum_{r,s=1}^{p} \left(\frac{\partial \omega_r^j}{\partial x^s} + \frac{\partial \omega_s^j}{\partial x^r} \right) (x_0) \frac{dx^r}{dt}(t_0) \cdot \frac{dx^s}{dt}(t_0)$$

Finally,

$$\varphi(t) - \varphi(t_0) = \frac{1}{2}\varphi''(t_0)(t - t_0)^2 + \mathcal{O}((t - t_0)^2),$$

from where $\varphi(t) \geq \varphi(t_0)$, $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. But $\varphi(t_0) = f(x_0)$ and, for $t \geq t_0$, $f(\alpha(t)) \geq \varphi(t)$ so that there results $f(\alpha(t)) \geq f(x_0)$ for $t \in [t_0, t_0 + \varepsilon)$.

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