TRIANGULAR MAPS WITH THE CHAIN RECURRENT POINTS PERIODIC

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Abstract. Forti and Paganoni [Grazer Math. Ber. 339 (1999), 125–140] found a triangular map \( F(x, y) = (f(x), g_x(y)) \) from \( I \times I \) into itself for which closed set of periodic points is a proper subset of the set of chain recurrent points. We asked whether there is a characterization of triangular maps for which every chain recurrent point is periodic. We answer this question in positive by showing that, for a triangular map with closed set of periodic points and any positive real \( \varepsilon \), every \( \varepsilon \)-chain from a chain recurrent point to itself may be represented as a finite union of \( \varepsilon \)-chains whose all points either are periodic or form a nontrivial \( \varepsilon \)-chain of some one-dimensional map \( g_x \).

1. Introduction and the theorem

Denote by \( I \) the closed interval \([0, 1] \subseteq \mathbb{R}\) with the induced topology, by \( I^2 \) the Cartesian product \( I \times I \subseteq \mathbb{R}^2 \), by \( X \) an arbitrary compact metric space. If \( A, B \subseteq X \) then \( \overline{A}, \text{int}(A) \) and \( \text{dist}(A, B) \) is the closure, the interior of \( A \) and the distance of \( A \) and \( B \), respectively.

The set of continuous mappings of a compact metric space \( X \) into itself is denoted by \( C(X, X) \). For \( \varphi \in C(X, X) \) and \( x \in X \), define inductively the \( n \)th iteration of \( \varphi \) by \( \varphi^0(x) = x \) and \( \varphi^n(x) = \varphi(\varphi^{n-1}(x)) \); \( n \) is a member of positive integer set \( \mathbb{N} \). The orbit \( \text{Orb}(A) \) of a set \( A \subseteq X \) is the set of its images under \( \varphi \). The trajectory of \( x \) is the sequence \( \{\varphi^n(x)\}_{n=0}^{\infty} \), and the set \( \omega_\varphi(x) \) of limit points of the trajectory of \( x \) is the \( \omega \)-limit set of \( x \). Let \( \omega(\varphi) = \bigcup_{x \in I} \omega_\varphi(x) \).

The \( \alpha \)-limit point of \( x \) is the limit point of some sequence \( \{x_n\}_{n=0}^{\infty} \) such that \( x_0 = x \) and \( \varphi(x_n) = x_{n-1} \). Let \( \alpha_\varphi(x) \) be the set of all \( \alpha \)-limit points of \( x \) and let \( \text{Fix}(\varphi) = \{x \in I : \varphi(x) = x\} \) be the set of fixed points of \( \varphi \). A point \( x \) is a periodic point of \( \varphi \), if \( \varphi^p(x) = x \) for some \( p \in \mathbb{N} \). The orbit of a periodic point \( x \) is a cycle and its cardinality is its period. Denote by \( P(\varphi) \) the set of periodic points of \( \varphi \).

A point \( x \) is nonwandering if for every open neighborhood \( U \) of \( x \) there is an \( n \in \mathbb{N} \) such that \( \varphi^n(U) \cap U \neq \emptyset \). Let \( \Omega(\varphi) \) denote the set of nonwandering points of \( \varphi \).

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Let $\varepsilon > 0$ be given and let $x, y$ be points of $X$. An $\varepsilon$-chain from $x$ to $y$ is a finite sequence $\{x_0, x_1, \ldots, x_n\}$ of points of $X$ with $x = x_0$, $y = x_n$ and $d_X(x_{k-1}, x_k) < \varepsilon$ for $k = 1, 2, \ldots, n$. An $\varepsilon$-chain $E$ is nontrivial if at least one point of $E$ is nonperiodic. A point $x$ is chain recurrent if and only if, for every $\varepsilon > 0$, there is an $\varepsilon$-chain from $x$ to itself ($x \rightarrow_{\varepsilon} x$). Let $CR(\varphi)$ denote the set of all chain recurrent points of $\varphi$.

The relations between the above-mentioned sets are given by the next proposition which can be found, e.g., in [1].

**Theorem 1.1.** If $X$ is a compact metric space and $\varphi \in C(X, X)$ then

(1) \[ \text{Fix}(\varphi) \subseteq P(\varphi) \subseteq \omega(\varphi) \subseteq \Omega(\varphi) \subseteq CR(\varphi). \]

The mentioned sets are equal if $f \in C(I, I)$ satisfies some assumptions; a long list of such assumptions can be found, e.g., in [4]. The next proposition which can be found, e.g., in [3], displays a few of them.

**Theorem 1.2.** [3] $f \in C(I, I)$. Then the following conditions are equivalent:

(i) $P(f) = \overline{P(f)}$.
(ii) $P(f) = \Omega(f)$.
(iii) $P(f) = CR(f)$.

Our paper is devoted to triangular maps of the square, i.e., continuous maps $F : I^2 \rightarrow I^2$ of the form $F(x, y) = (f(x), g(y))$; the function $f$ is called the base function of $F$. Denote by $T(I^2, I^2)$ the set of triangular maps $I^2 \rightarrow I^2$ and by $d$ the max metric on $I^2$. In 1990 L. S. Efremova proved the next result (in a more strong form).

**Theorem 1.3.** [2] Let $F \in T(I^2, I^2)$ and $P(F)$ be closed. Then $P(F) = \Omega(F)$.

Let (i)–(iii) be the properties displayed in Proposition 1.2 considered for a triangular map $F$. Since the set $\Omega(F)$ is closed, the Proposition 1.3 and (1) give (i)$\Leftrightarrow$(ii).

Moreover, Forti and Paganoni in [3] found a triangular map $F$ with closed set of the periodic points such that $CR(F) \setminus \omega(F) \neq \emptyset$. By (1), $\omega(F) \subseteq \Omega(F) \subseteq CR(F)$, so there is a problem to characterize triangular maps $F$ with closed set $P(F)$ such that $P(F) = CR(F)$. The wanted characterization follows from the next theorem which is the main result of our paper.

**Definition 1.4.** Let $z_1 = (x, y_1), z_2 = (x, y_2)$ be periodic points of $F$. The point $z_1$ is accessible from $z_2$ ($z_2 \rightarrow^a z_1$) if there is an $\varepsilon$-chain from $z_2$ to $z_1$ by the map $F$ restricted to $I_{Orb}(x) = \bigcup_{y \in Orb(z_2)} \{y\} \times I$ for any $\varepsilon > 0$. The point $z_1$ is nontrivially accessible from $z_2$ ($z_2 \rightarrow^a z_1$) if $z_2 \rightarrow^a z_1$ and for any sufficiently small $\varepsilon > 0$ any $\varepsilon$-chain from $z_2$ to $z_1$ by the map $F$ restricted to $I_{Orb}(x)$ is nontrivial.

Let $K_1, K_2$ be subsets of $P(F)$. Then $K_1$ is accessible from $K_2$ ($K_2 \rightarrow^a K_1$) if $z_2 \rightarrow^a z_1$ for some $z_1 \in K_1$ and $z_2 \in K_2$.

**Definition 1.5.** Points $z_1, z_2 \in P(F)$ form a t-pair if (i) $z_2 \rightarrow^a z_1$ and, for any $\delta > 0$, (ii) there exists a finite number of connected components $K_i$. 
Let $F \in T(I^2, I^2)$ and $P(F)$ be closed. Then $CR(F) \setminus P(F) \neq \emptyset$ if and only if there exists a $t$-pair.

2. KNOWN FACTS

For a map $F \in T(I^2, I^2)$, let $A(F)$ denote any of the sets $\Fix(F), P(F), \omega(F), \Omega(F)$ and $CR(F)$ and let $\pi$ be the canonical projection $(x, y) \mapsto x$.

**Theorem 2.1.** [4] Let $F \in T(I^2, I^2)$, let $f$ be the base function of $F$. Then $\pi(A(F)) = A(f)$.

We already know that $CR(\varphi)$ is closed for any $\varphi \in C(X, X)$. Of course, $CR(\varphi)$ enjoy a number of other general properties used later.

**Theorem 2.2.** [1] Let $\varphi \in C(X, X)$. Then, for any $n \in \mathbb{N}$, $CR(\varphi) = CR(\varphi^n)$ and $P(\varphi) = P(\varphi^n)$.

**Theorem 2.3.** [1] Let $\varphi \in C(X, X)$. Then $CR(\varphi) = CR(\varphi|_{CR(\varphi)})$, i.e. every chain recurrent point remains chain recurrent for the restriction of $\varphi$ to $CR(\varphi)$.

Let $\varphi \in C(X, X)$. A nonempty closed set $A$ contained in $X$ is Lyapunov stable if, for each open set $U$ containing $A$, there exists an open set $V$ containing $A$ such that $\varphi^n(V) \subseteq U$ for all $n \in \mathbb{N}$. A nonempty closed set $A$ is an attractor if there exists an open set $B$ containing $A$ such that $\omega_\varphi(x) \subseteq A$ for every $x \in B$. If a nonempty closed set $A$ is both Lyapunov stable and an attractor, we say that $A$ is asymptotically stable.

**Theorem 2.4.** [1] Let $\varphi \in C(X, X)$. If $A \subseteq X$ is an asymptotically stable set, then there exists an open set $W$ containing $A$ such that $\varphi(W) \subseteq W$. Moreover, for any open upset $U$ of $A$ we can choose $W$ so that $\overline{W} \subseteq U$.

Denote by $Q(x, \varphi)$ the intersection of all asymptotically stable sets containing $\omega_\varphi(x)$.

**Theorem 2.5.** [1] Let $\varphi \in C(X, X)$. If $y \in Q(x, \varphi)$, then $Q(y, \varphi) \subseteq Q(x, \varphi)$.

**Theorem 2.6.** [1] Let $\varphi \in C(X, X)$. If $y \in Q(x, \varphi)$, then there is an $\varepsilon$-chain from $x$ to $y$ by $\varphi$ for any $\varepsilon > 0$.

3. LEMMAS AND PROOFS

Let $\varphi \in C(X, X)$ and $K, L \subseteq X$. Denote by $S_\varphi(K, L)$ the closure of $\{ z \in X | z$ is a member of an $\varepsilon$-chain from a point of $K$ to a point of $L$ by $\varphi \}$. If $K \rightarrow_\varepsilon L$ for any $\varepsilon > 0$, the set $S_\varphi(K, L) = \bigcap_{\varepsilon > 0} S_\varphi(K, L)$ is clearly nonempty and closed. If $z_0 \in CR(\varphi)$, the set $S_\varphi(z_0, z_0)$ is the independent set of $z_0$.

**Lemma 3.1.** Let $\varphi \in C(X, X)$ and $z_0 \in CR(\varphi)$. Then the independent set $S := S_\varphi(z_0, z_0)$ is strongly invariant, i.e. $\varphi(S) = S$. 

Proof. This easily follows from the continuity of $\varphi$ and the definition of $S$. 

**Lemma 3.2.** Let $\varphi \in C(X, X)$ and $z_0 \in CR(\varphi)$. Then any point of the independent set $S := S_{\varphi}(z_0, z_0)$ is chain recurrent for the restriction of $\varphi$ to $S$, i.e. $S = CR(\varphi|_S)$.

Proof. It is easy to see, $CR(\varphi|_S) \subseteq S$. We prove the converse inclusion.

Clearly $S \subseteq CR(\varphi)$. Let $z$ be a chain recurrent point lying in $S$. At first we show that for any open neighborhood $U$ of $S$ and any $\varepsilon > 0$ there is an $\varepsilon$-chain from $z$ to $z$ in $U$. Assume, contrary to what we wish to show, that there exists an open neighborhood $U$ of $S$ and $\varepsilon > 0$ such that every $\varepsilon$-chain from $z$ to itself contains a point in the complement of $U$. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers tending to 0. Then, for each $n \in \mathbb{N}$, there is a point $z_n$ of an $\varepsilon_n$-chain lying outside $U$. Denote by $\bar{z}$ an accumulation point of $\{z_n\}_{n \in \mathbb{N}}$. Evidently, $\bar{z} \notin U$. But $z \rightarrow^\varepsilon z$ and $\bar{z} \rightarrow^\varepsilon z$ for every $\varepsilon > 0$ and, by the definition of $S$ and $z \in S$, this implies $\bar{z} \in S$, which is a contradiction.

It remains to show that for any $\varepsilon > 0$ there is an $\varepsilon$-chain from $z$ to itself within $S$. By the continuity of $\varphi$, we can choose $\delta > 0$, $0 < \delta < \frac{\varepsilon}{3}$, so that $d_X(x, y) < \delta$ implies $d_X(\varphi(x), \varphi(y)) < \frac{\varepsilon}{3}$. Let $U$ be an open $\delta$-neighborhood of $S$ and let $\{z_0, z_1, \ldots, z_k\}$ be an $\frac{\varepsilon}{3}$-chain from $z$ to $z$ lying in $U$. There exist points $s_0, s_1, \ldots, s_k$ in $S$ with $s_0 = s_k = z$ and $d_X(z_j, s_j) < \delta$ for $1 \leq j \leq k$. Then for each $j = 1, 2, \ldots, k$

$$d_X(\varphi(s_{j-1}), s_j) < d_X(\varphi(s_{j-1}), \varphi(z_{j-1})) + d_X(\varphi(z_{j-1}), z_j) + d_X(z_j, s_j) < \varepsilon,$$

which already proves $S = CR(\varphi|_S)$. 

**Lemma 3.3.** Let $F \in T(I^2, I^2)$, $P(F)$ be closed and $z = (x, y) \in CR(F) \setminus P(F)$. If $x$ is a fixed point of the base map $f$, then the independent set $S_f(x, x)$ is a non-degenerate closed interval of $P(f)$.

Proof. Let the assumptions be fulfilled. Clearly,

$$(2) \quad \pi(S_F(z, z)) \subseteq S_f(x, x).$$

Denote $S := S_F(z, z)$ and $D := S_f(x, x)$. Since any independent set is closed, strongly invariant and the fixed point $x$ belongs to $D$, $D$ is closed interval.

We show that $D$ is nondegenerate. Assume, on the contrary, $D = \{x\}$. By (2), $S \subseteq \{x\} \times I$ and hence, with respect to Lemma 3.2, $y \in CR(g_x)$ since $z \in S$. On the other hand, $z \notin P(F)$ implies $y \notin P(g_x)$, i.e. $y \in CR(g_x) \setminus P(g_x)$. So, by Proposition 1.2, $P(g_x)$ is not closed, which contradicts $P(F) = P(F)$.

Finally, we show $D \subseteq P(f)$. By Lemma 3.2, any point of $D$ is chain recurrent. Recall that $P(F)$ is closed and the projection of any closed set is closed. Hence, by Propositions 1.2 and 2.1, $D \subseteq P(f)$.

**Lemma 3.4.** Let $F \in T(I^2, I^2)$, $P(F)$ be closed and $f$ be the base map of $F$. Let $z = (x, y) \notin P(F)$ be such that $x \in Fix(f)$. If any $z_1 \in \omega_F(z)$, $z_2 \in \alpha_F(z)$ lie in the same connected component of $P(F)$, then $z_1, z_2$ form a t-pair. Otherwise $z_2 \rightarrow^\alpha z_1$. 


Proof. Let the assumptions be fulfilled. Clearly, \( z_1 = (x, y_1), z_2 = (x, y_2) \) are periodic points of \( F \). We show that \( y_1 \neq y_2 \). Really, by the choice of \( z_1, z_2 \), for any \( \varepsilon > 0 \), \( y \to_{\varepsilon} y_1 \) and \( y_2 \to_{\varepsilon} y \) by \( g_x \). Hence \( y_1 = y_2 \) implies \( y \in CR(g_x) \) and, accordingly, \( CR(g_x) \setminus P(g_x) \neq \emptyset \). So, by Proposition 1.2, \( P(g_x) \) is not closed, a contradiction.

It is easy to see that \( z_2 \to^n z_1 \). Hence \( z_1, z_2 \) form a t-pair if they lie in the same connected component of \( P(F) \). In the opposite case, \( z_2 \to^n z_1 \).

Proof of the main theorem. Let \( F \) be a triangular map and \( f \) its base map. The existence of a t-pair clearly implies the existence of nonperiodic chain recurrent point. Let us prove the converse implication.

Let \( z_0 = (x_0, y_0) \) be a nonperiodic chain recurrent point, \( S := S_F(z_0, 0) \) and \( D := S_f(x_0, x_0) \). We show that there is a t-pair in \( S \). Clearly, any t-pair of some iteration of \( F \) is t-pair of \( F \). Moreover, by Proposition 2.2, \( z_0 \in CR(F^n) \setminus P(F^n) \) for any \( n \in \mathbb{N} \). So we may consider some convenient iteration of \( F \) instead of \( F \). Thus we may assume \( z_0 \in Fix(f) \), since, by Propositions 1.2 and 2.1, \( x_0 \in P(f) \). Obviously, \( S \subseteq D \times I \), where, by Lemma 3.3, \( D \) is a closed nondegenerate interval of \( P(f) \). But any interval of \( P(f) \) containing a fixed point has only periodic points of period \( \leq 2 \). So we find more convenient iteration of \( F \) and may assume \( \pi(S) \subseteq Fix(f) \). Moreover, by Lemma 3.2, \( z_0 \in CR(F|_S) \setminus P(F|_S) \).

In the rest of the proof put \( F := F|_S \) and \( P := P(F) \setminus S \). Since the complement of \( P \) is open, there is a \( \delta > 0 \) such that an open \( \delta \)-neighborhood of \( z_0 \) contains no periodic point. Since \( S \) is strongly invariant, \( P \) is nonempty and hence we can define a relation \( \sim := \sim (\delta) \) on the set of all connected components of \( P \) in the following way: \( K \sim L \) if and only if \( K \to_\delta L \) by \( F \) restricted to \( P \). It is easy to see that \( \sim \) is an equivalence relation on \( P \). Members of the decomposition \( P|_\sim \) are \( \delta \)-components. The distance \( d(v, w) \geq \delta \) if \( v \in K, w \in L \) are points of distinct \( \delta \)-components \( K, L \) and hence, by the compactness of \( S \), there is only a finite number of \( \delta \)-components \( H_i \), \( i = 1, 2, \ldots, k \).

Lemma 3.5. Let \( H \) be a \( \delta \)-component and \( u, w \in H \). Then there are connected components \( L_1, L_2, \ldots, L_r \) of \( H \) such that \( u \in L_1, w \in L_r \) and \( dist(L_i, L_{i+1}) < \delta \) for every \( i = 1, 2, \ldots, r - 1 \).

Proof. Obvious. \( \square \)

By Lemma 3.1, \( \omega_F(z_0) \subseteq P \) and \( \alpha_F(z_0) \cap P \neq \emptyset \) since \( z_0 \in S \). Choose \( z_1 \in \omega_F(z_0) \). We show that there is a \( z_2 := z_2(\delta) \in \alpha_F(z_0) \cap P \) such that \( z_1 \) and \( z_2 \) satisfy the condition (ii) of Definition 1.5 \((z_2 \to_\delta^{i-1} z_1)\). With respect to Lemma 3.4, we may assume \( z_2 \to^n z_1 \). By Lemma 3.5, it is sufficient to find \( \delta \)-components \( K_1, K_2, \ldots, K_m \) such that \( z_1 \in K_1, z_2 \in K_m \) and

\[
(3) \quad K_i \to^n K_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, m - 1.
\]

Without loss of generality, assume \( z_1 \in H_1 \). If \( \alpha_F(z_0) \cap H_1 \neq \emptyset \), by Lemmas 3.4 and 3.5, \( z_2 \to_\delta^{i-1} z_1 \) for every \( z_2 \) from this intersection. So let \( \alpha_F(z_0) \cap (P \setminus H_1) \neq \emptyset \) and \( K_1 := H_1 \). Clearly \( K \to^n L \) implies \( \pi(K) \cap \pi(L) \neq \emptyset \). Hence we may restrict
our attention to the set $S_1 := S \cap (\pi(K_1) \times I)$ to show that there is a $\delta$-component $H$ such that $K_1 \to^{a} H$.

**Lemma 3.6.** $K_1 \to^{a} H$ for some $H \in P|_{\sim}$.

**Proof.** Assume, on the contrary, that

\begin{equation}
K_1 \not\to^{a} H \text{ for each } H \in P|_{\sim} \setminus K_1. 
\end{equation}

Fix $x \in \pi(S_1) \subseteq \text{Fix}(f)$. By Proposition 2.6, $Q(y, g_z) \cap (P \setminus K_1) = \emptyset$ for any $(x, y) \in K_1(x) := K_1 \cap I_x$. Hence $Q(x) \cap P = K_1$, where $Q(x)$ denotes $\bigcup_{(x, y) \in K_1(x)} Q(y, g_z)$. It is well known fact, that $Q(\cdot, \cdot)$ is an asymptotically stable set. Generally, only a finite union of asymptotically stable sets must be asymptotically stable, see [1]. Consequently, it suffices to clarify that $Q(x)$ is "generated" by a finite number of points of $K_1(x)$ to show that $Q(x)$ is asymptotically stable.

Since $P(F)$ is closed, $K_1$ is closed and hence, any sequence $\{y_n\} \subseteq K_1$ has an limit point $y \in K_1(x)$. Because $Q(y, g_z)$ is asymptotically stable, there exists $N \in \mathbb{N}$ such that $y_n \in Q(y, g_z)$ for every $n \geq N$. Moreover, if $J$ is subinterval of $K_1(x)$, $J \subseteq Q(y, g_z)$ for any $(x, y) \in J$. By these two facts, Proposition 2.5 and the compactness of $K_1(x)$, $Q(x)$ is asymptotically stable.

Accordingly, by (4), $Q(x)$ is asymptotically stable for any $x \in \pi(S_1)$. Since we have restricted $F$ to $S_1$, there is, by Proposition 2.4, an open set $W$ containing $K_1$ such that

\begin{equation}
F(W) \subseteq W. 
\end{equation}

Clearly, again by Proposition 2.4, we may assume $z_0 \notin W$. But we have shown that $\omega_f(z_0) \subseteq K_1$. Hence, for a sufficiently small $\varepsilon > 0$, some point of any $\varepsilon$-chain from $z_0$ to itself lies in $W$. On the other hand, since $W$ is a subset of any open neighborhood of $K_1$, by (5), there is no $\varepsilon$-chain from $H_1$ to $z_0$ for some sufficiently small $\varepsilon > 0$, a contradiction. □

Denote $H'_1 := K_1$. By Lemma 3.6, a set $H'_2 := \{H \in P|_{\sim} \setminus \left( H'_1 \mid K_1 \to^{a} H \right) \}$ is nonempty. If $\alpha_f(z_0) \cap H'_2 \neq \emptyset$, by Lemmas 3.4 and 3.5, there is a $z_2 := z_2(\delta) \in \alpha_f(z_0) \cap H'_2$ such that $z_2 \to^{\delta} z_1$. If $\alpha_f(z_0) \cap H'_2$ is empty, we analogously define the set $H'_3 := \{H \in P|_{\sim} \setminus \left( H'_1 \cup H'_2 \right) \ | \exists L \in H'_2 : L \to^{a} H \}$ and similarly show that $H'_3 \neq \emptyset$. Accordingly, since the cardinality of $P|_{\sim}$ is finite, there are nonempty sets $H'_1, H'_2, \ldots, H'_m$, $m \leq k$, of $\delta$-components of $P$ such that $\alpha_f(z_0) \cap K \neq \emptyset$ for some $\delta$-component $K \in H'_m$ and, for any $j \in \{1, 2, \ldots, m-1\}$, any member of $H'_j$ is accessible from some member of $H'_j$.

This already proves that for any $\delta > 0$ there is a $z_2 := z_2(\delta) \in \alpha_f(z_0) \cap S$ such that $z_2 \to^{\delta} z_1$. It is easy to see that $z_2(\delta) \to^{\tau} z_1$ for any $\tau > \delta$. Accordingly, for any decreasing sequence of positive reals $\{\delta_i\}$ tending to 0 the limit point $\hat{z}_2$ of the sequence $\{z_2(\delta_i)\}$ and $z_1$ is a t-pair. □
References


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