MIC(0) DD PRECONDITIONING OF FEM ELASTICITY PROBLEMS ON NON-STRUCTURED MESHES

SVETOZAR MARGENOV† AND PETER POPOV ‡

Abstract. Two preconditioning techniques for PCG iterative solution of 2D elasticity problems are described and studied. Linear and quadratic Lagrangian finite elements are applied for discretization of the elliptic system of partial differential equations. Non-structured meshes, including a local refinement, are used for triangulation of the computational domain. The obtained coupled stiffness matrix has a two-by-two block structure corresponding to a simple separable displacement ordering of the unknowns. Then, two displacement decomposition (DD) algorithms are considered whereas MIC(0) incomplete factorization is used to approximate the related diagonal blocks. A set of numerical experiments are presented to illustrate the PCG convergence rate of the related MIC(0) DD preconditioners.

Key words. PCG, MIC(0), displacement decomposition, FEM

AMS subject classifications. 65F10, 65N30

1. Introduction. The performance of two displacement decomposition (DD) preconditioners that exploit modified incomplete Cholesky factorization MIC(0) is studied in the case of finite element (FEM) matrices arising from the discretization of the two-dimensional (2D) equations of elasticity. There is a lot of work dealing with preconditioning iterative solution methods for the Navier equations of elasticity. Here we will briefly comment on some of the approaches used. Multigrid, multilevel, domain decomposition, and patched local refinement are directly applicable in a very general setting to the coupled stiffness matrix. Another example of this first approach is the block-ILU factorization based on block-size reduction ([6], see also in [7, 8]).

In this paper we focus our attention on a different preconditioning strategy based on a first step block-diagonal approximation of the stiffness matrix written in a separable displacement two-by-two block form. In an earlier paper, Axelsson and Gustafsson [3] have implemented modified point-ILU factorization to this problem. As the coupled system does not lead to an $M$-matrix, they construct their preconditioners based on the point-ILU factorization of the displacement decoupled (displacement decomposition) block-diagonal part of the original matrix. This approach is based on the second Korn's inequality, providing good convergence of the resulting preconditioned conjugate gradient (PCG) algorithm for elasticity problems with moderate Poisson ratio $\nu \in [0, \frac{1}{2})$ (relatively far from the incompressible limit $\nu = \frac{1}{2}$).

The displacement decomposition remains up to now one of the most robust approaches for preconditioning of FEM elasticity systems (see also, e.g., [5, 13]). The most often studied case is when the mesh is (topo)logically equivalent to uniform one in the unit square/cube (see, e.g., [3, 5, 10, 6, 12, 10]), or more generally, when a patched local refinement is additionally applied [7]. The problem considered here is in a general setting. The computational domain is arbitrary where the triangulation is

---

*This work was supported partially by Grant 1801/98.
†Central Laboratory for Parallel Processing, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 25A, 1113 Sofia, Bulgaria, (margenov@parallel.bas.bg).
‡Aimspace Engendering Department, Texas A&M University, College Station, TX 77843, USA, (ppopov@tamu.edu)
automatically generated including subdomains with local refinement. Discretization by linear and quadratic Lagrangian triangle finite elements is considered. As the diagonal blocks of the displacement decomposed matrix are not $M$-matrices in the case of quadratic elements, the $MIC(0)$ preconditioner is applied to the related linear FEM stiffness matrices defined on an auxiliary globally refined mesh.

The aim of the presented numerical tests is to analyze the performance of the studied algorithms. Our consideration is focused on the scalability of the PCG convergence rate where the mutually interacted impacts of the higher order $FEM$, the non-structured grids including local refinement, the different kinds of singularities of the solution and the coefficients jumps are compared.

The reminder of the paper is organized as follows: In section 2 we survey some major results about the Navier equations of elasticity and their $FEM$ approximation. The needed background results related to the $DD$ $MIC(0)$ preconditioner are presented in section 3. Section 4 contains a rich set of numerical experiments organized in two subsections. The first of them is dedicated to the asymptotic behaviour of the studied iterative solvers based on a model test problem. The second one contains real-life benchmark problems on non-structured meshes including local mesh refinement. Short concluding remarks are given in the last section.

2. FEM approximation of the Navier equations of elasticity. Let us consider the weak formulation of the linear elasticity problem in the form: find $\mathbf{u} \in V_E = [H^2(\Omega)]^3 = \{ \mathbf{v} \in [H^1(\Omega)]^3 : \mathbf{u}_G = \mathbf{u}_s \}$ such that

$$
\int_\Omega [2\mu \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + \lambda \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}] \, d\Omega = \int_\Omega \mathbf{f} : \mathbf{v} \, d\Omega + \int_{\Gamma_N} \mathbf{g}_n \mathbf{n} \cdot \mathbf{v} \, d\Gamma_N,
$$

for all $\mathbf{v} \in V_0 = [H^1(\Omega)]^3 = \{ \mathbf{v} \in [H^1(\Omega)]^3 : \mathbf{u}_G = 0 \}$, with the positive parameters $\lambda$ and $\mu$ of Lamé, the symmetric strains

$$
\varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T),
$$

the volume forces $\mathbf{f}$, and the boundary tractions $\mathbf{g}$. Usually one introduces Young's modulus $E > 0$ and Poisson's ratio $\nu \in [0, 1)$. The relations between these and the material parameters $\lambda, \mu$ are

$$
\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}.
$$

Assume that $\Omega$ is a polygonal domain, $\mathcal{T}$ is a triangulation of $\Omega$, and let for simplicity $E$ and $\nu$ be piecewise constants with respect to the elements of $\mathcal{T}$. Now we denote by $V_E^{(\mathcal{T}, k)} \subset V_E$ and $V_0^{(\mathcal{T}, k)} \subset V_0$ the FEM spaces of conforming piecewise linear and piecewise quadratic ($k \in \{1, 2\}$) functions with nodal Lagrangian basis corresponding to the triangulation $\mathcal{T}$ with a mesh parameter $h$. Applying FEM discretization to (1) we get the linear algebraic system

$$
K^{(h, k)} \mathbf{u}_h = \mathbf{f}_h
$$

where $K^{(h, k)}$ is the related symmetric and positive definite stiffness matrix and $\mathbf{u}_h$ is the vector of nodal unknowns. Let us assume that $K^{(h, k)}$ is written in a two-by-two block form

$$
K^{(h, k)} = \begin{pmatrix}
K_{11}^{(h, k)} & K_{12}^{(h, k)} \\
K_{21}^{(h, k)} & K_{22}^{(h, k)}
\end{pmatrix},
$$

where

$$
K_{ij}^{(h, k)} = \int_{\Omega} \varepsilon(\mathbf{u}_i) : \varepsilon(\mathbf{v}_j) \, d\Omega + \int_{\Gamma_N} \gamma_{ij} \mathbf{n} \cdot \mathbf{v}_j \, d\Gamma_N,
$$

for $i, j \in \{1, 2\}$, $\gamma_{ij} \in \mathbb{R}$ the $k$-th order element parameters.
where the block structure corresponds to a separable displacement component ordering of the vector of nodal unknowns. The aim of our study is to develop robust preconditioning solution methods for the system (3) suitable for large-scale problems. For a block-approximation of $K^{(h,k)}$ we use two displacement decomposition approaches, see e.g. [3, 5], referred as separable displacement preconditioner $C^{(h,k)}_{SDC}$ and isotropic preconditioner $C^{(h,k)}_{ISO}$ defined as follows

$\begin{align}
C^{(h,k)}_{SDC} &= \begin{pmatrix} K^{(h,k)}_{11} & K^{(h,k)}_{12} \\ K^{(h,k)}_{21} & K^{(h,k)}_{22} \end{pmatrix} \\
C^{(h,k)}_{ISO} &= \begin{pmatrix} \bar{K}^{(h,k)}_{11} & \bar{K}^{(h,k)}_{12} \\ \bar{K}^{(h,k)}_{21} & \bar{K}^{(h,k)}_{22} \end{pmatrix}
\end{align}$

where $\bar{K}^{(h,k)} = (K^{(h,k)}_{11} + K^{(h,k)}_{22})/2$. Such preconditioning strategies of the coupled matrix $\bar{K}$ are theoretically motivated by the second Korn’s inequality. More precisely, the next lemma holds.

**Lemma 2.1.** The displacement decomposition preconditioners (4, 5) are of optimal order, that is,

$\text{cond}((C^{(h,k)}_{SDC})^{-1} K^{(h,k)}) = O(1 - 1 - 2\nu_{\max}), \quad \text{cond}((C^{(h,k)}_{ISO})^{-1} K^{(h,k)}) = O(1 - 1 - 2\nu_{\max})$

where $\nu_{\max} = \max_{\Omega} \nu$.

The above estimates straightforwardly follow from second Korn’s [5, 11] inequality for the continuous elasticity problem, and from the inclusion $V_0^{(h,k)} \subset V_0$. The constants in the estimates depend on the Poisson’s ratio.

**Remark 1.** In the model case of pure displacement problem of homogeneous material the exact estimates read as follows [3]:

$\text{cond}((C^{(h,k)}_{SDC})^{-1} K^{(h,k)}) < \frac{2}{1 - \bar{\nu}}, \quad \text{cond}((C^{(h,k)}_{ISO})^{-1} K^{(h,k)}) < \frac{3 + \bar{\nu}}{1 - \bar{\nu}},$

where $\bar{\nu} = \frac{\nu}{1 - \nu} \in [0, 1)$. Some interesting generalizations of these estimates are presented in the more recent paper [10].

**3. DD MIC(0) preconditioning.** In this section we first recall some facts about the modified incomplete Cholesky factorization MIC(0) preconditioning algorithm. Our presentation at this point follows those from [5]; see also [13] for an alternative approach. Let us rewrite the real $N \times N$ symmetric matrix $A = (a_{ij})$ in the form

$\begin{align}
A &= D - L - L^t \\
\end{align}$

where $D$ is the diagonal and $(-L)$ is the strictly lower triangular part of $A$. Then we consider the following approximate factorization of $A$:

$C_{\text{MIC}(0)} = (X - L)X^{-1}(X - L)^t$

where $X = \text{diag}(x_1, \ldots, x_N)$ is a diagonal matrix determined by the condition of equal rowsums, i.e.,

$C_{\text{MIC}(0)} \varepsilon = A \varepsilon, \quad \varepsilon = (1, \ldots, 1)^t \in \mathbb{R}^N$
For the purpose of preconditioning, we are interested in the case when $X > 0$ and thus $C_{MIC(0)}$ is positive definite. If this holds, we speak about stable MIC(0) factorization. Concerning stability of the MIC(0) factorization, we have the following theorem.

**Theorem 3.1.** Let $A = (a_{ij})$ be a symmetric real $N \times N$ matrix and let $A = D - L - L^t$ be the splitting (6) of $A$. Let us assume that

$$
L \succeq 0, \\
A \succeq 0, \\
A + L' \succ 0 \quad \epsilon = (1, \ldots, 1) \in \mathbb{R}^N,
$$

i.e. that $A$ is a weakly diagonally dominant matrix with nonpositive offdiagonal entries and that $A + L' = D - L$ is strictly diagonally dominant.

Then the relation

$$
x_i = a_{ii} - \sum_{k=1}^{i-1} a_{ik} \sum_{j=k+1}^{N} a_{kj}
$$

gives the positive values $x_i$ and the diagonal matrix $X = \text{diag}(x_1, \ldots, x_N)$ defines stable MIC(0) factorization of $A$.

The conditions of the theorem are obviously not always satisfied for FEM stiffness matrices. Now we introduce the displacement decomposition preconditioners for the case of linear FEM ($k = 1$) in the form

$$
C_{SDC\, MIC(0)}(K^{(h,1)}) = \begin{pmatrix}
C_{MIC(0)}(K^{(h,1)}_{11}) \\
C_{MIC(0)}(K^{(h,1)}_{22})
\end{pmatrix},
$$

$$
C_{SDC\, ISO(0)}(K^{(h,1)}) = \begin{pmatrix}
C_{MIC(0)}(K^{(h,1)}) \\
C_{MIC(0)}(R^{(h,1)})
\end{pmatrix}.
$$

Here $C_{MIC(0)}(A)$ stands for the MIC(0) preconditioner of $A$. A local elementwise analysis of the MIC(0) stability is applicable in this case. Since the blocks $K^{(h,1)}_{11}$ and $K^{(h,1)}_{22}$ correspond to elliptic bilinear forms they can be assembled by element stiffness matrices written in a general form (see for more details in [1, 4])

$$
A^{(h,1)}_T = r_T \begin{pmatrix}
b_T + c_T & -c_T & -b_T \\
-c_T & a_T + c_T & -a_T \\
-b_T & -a_T & a_T + b_T
\end{pmatrix}
$$

where $a_T = \cot \alpha_T$, $b_T = \cot \beta_T$, $c_T = \cot \gamma_T$, $(\alpha_T, \beta_T, \gamma_T)$ can be viewed as angles of some triangle, and $r_T$ is a constant, all generally depending on both the shape of $T \in \mathcal{T}$ and on the Poisson’s ratio $\nu$. The next lemma follows from (9).

**Lemma 3.2.** If the corresponding angles $(\alpha_T, \beta_T, \gamma_T)$ satisfy the condition

$$
\max_{e \in T} \max \{\alpha_T, \beta_T, \gamma_T\} \leq \frac{\pi}{2}
$$

then MIC(0) factorization of the blocks $K^{(h,1)}_{11}$, $K^{(h,1)}_{22}$ and $K^{(h,1)}$ is stable.

The condition (10) is easily controlled in the case of moderate Poisson’s ratio by a proper choice of the minimal angle of the triangulation which is one of the options of the used general purpose mesh generator.
In the case of quadratic FEM discretization the diagonal blocks do not belong to the class of $M$-matrices even for the model problem $-\Delta u = f$. To get a stable $MIC(0)$ factorization, in this case, we first globally refine the triangulation $T$ by connecting the midpoints of the triangles sides and denote by $K_{11}^{(h/2,1)}$, $K_{22}^{(h/2,1)}$ and $K_{12}^{(h/2,1)}$ the related blocks corresponding to linear FEM implemented on the refined mesh. Then the $MIC(0)$ displacement decomposition preconditioners are defined as follows:

\[
C_{SDC\_MIC(0)}(K^{(h,2)}) = \begin{pmatrix} C_{MIC(0)}(K_{11}^{(h/2,1)}) & 0 \\ 0 & C_{MIC(0)}(K_{22}^{(h/2,1)}) \end{pmatrix}
\]

\[
C_{SDC\_ISO(0)}(K^{(h,2)}) = \begin{pmatrix} C_{MIC(0)}(R_{11}^{(h/2,1)}) & 0 \\ 0 & C_{MIC(0)}(R_{22}^{(h/2,1)}) \end{pmatrix}.
\]

The next lemma provides the theoretical background of the above preconditioning approach.

**Lemma 3.3.** The following spectral equivalence estimates hold

\[\text{cond}((K_{ii}^{(h/2,1)})^{-1} K_{ii}^{(h,2)}) = O(1), \quad i \in \{1, 2\}.\]

The proof of the lemma can be done by a local element by element analysis similarly as for the model elliptic problem presented, e.g., in [1].

**Remark 2.** The numerical tests discussed in the last section are performed using the perturbed version of $MIC(0)$ algorithm, where the incomplete factorization is applied to the matrix $\hat{A} = A + D$. The diagonal perturbation $D = D(\xi) = \text{diag}(d_1, \ldots, d_N)$ is defined as follows:

\[
d_i = \left\{ \begin{array}{ll} \xi a_{ii} & \text{if } a_{ii} \geq 2w_i \\ \xi^{1/2} a_{ii} & \text{if } a_{ii} < 2w_i \end{array} \right.
\]

where $0 < \xi < 1$ is a constant of order $O(h^2)$ and $w_i = \sum_{j>i} -a_{ij}$.

**Remark 3.** It is well known that the convergence of the incomplete factorization preconditioning algorithms strongly depends on the ordering of the unknowns. The results presented in this paper correspond to a coordinatewise ordering of the nodes.

4. **Numerical experiments.** Two types of numerical experiments are reported in this section. In both cases, the number of iterations is used as a measure for the robustness of the iterative solvers.

4.1. **Model problem.** Consider the model pure displacement ($GD = \partial \Omega$) problem with homogeneous material ($\lambda = 1$, and $\mu = 1.5$), and right hand side corresponding to the given solution $u_1 = x^2 + \cos(x + y)$, $u_2 = x^3 + y^4 + \sin(y - x)$. A uniform mesh with $n$ mesh points along each of the coordinate directions is used where the discretization parameter $n \in \{4, 8, 16, 32, 64\}$ is varied. In Table 1 are presented number of iterations for five methods: the well known Gauss-Seidel ($G - S$), Steepest Decent ($SD$), and Conjugate Gradient ($CG$), and the studied in this paper two displacement decomposition $MIC(0)$ PCG methods. The stopping criteria here is $\|u_h^{(k+1)} - u_h\|_0 \leq 10^{-10} \|u_h\|_0$. The advantages of $SDC\_MIC(0)$ and $ISO\_MIC(0)$ are well expressed even for relatively small discrete problems. For the model problem $SDC\_MIC(0)$ converges a little bit faster, but as will be shown in the next subsection, this is not a general conclusion. The asymptotic of the number of iterations of both $DD\_MIC(0)$ preconditioners is $O(\sqrt{n})$. 
4.2. Real-life problems. Three real-life engineering problems are considered as benchmarks in this subsection. The computational tests are performed by the developed FEM integrated program system where the general purpose mesh generator TRIANGLE [14] has been incorporated. The numerical tests are presented in a table form where the number of iterations of CG, and PCG with the studied SDC MIC(0) and ISO MIC(0) preconditioners are listed for both linear and quadratic FEM with a relative stopping criteria in a form \( \|K_{h}^{(k+1)} - f_{h}\|_{0} < 10^{-8}\|f_{h}\|_{0} \).

<table>
<thead>
<tr>
<th>n</th>
<th>G - S</th>
<th>SD</th>
<th>CG</th>
<th>SDC</th>
<th>ISO</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>171</td>
<td>387</td>
<td>58</td>
<td>21</td>
<td>24</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>1455</td>
<td>96</td>
<td>26</td>
<td>30</td>
</tr>
<tr>
<td>16</td>
<td>231</td>
<td>5885</td>
<td>190</td>
<td>33</td>
<td>39</td>
</tr>
<tr>
<td>32</td>
<td>8822</td>
<td>18741</td>
<td>468</td>
<td>44</td>
<td>54</td>
</tr>
<tr>
<td>64</td>
<td>32299</td>
<td>76000</td>
<td>725</td>
<td>60</td>
<td>73</td>
</tr>
</tbody>
</table>

![Fig. 1. Benchmark 1. Concrete body with a crack.](image)

![Diagram](image)

Table 2

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Number of nodes</th>
<th>CG</th>
<th>SDC MIC(0)</th>
<th>ISO MIC(0)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear/Quadratic</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1009/6242</td>
<td>495</td>
<td>1254</td>
<td>107</td>
</tr>
<tr>
<td>2</td>
<td>2781/10833</td>
<td>729</td>
<td>1876</td>
<td>147</td>
</tr>
<tr>
<td>3</td>
<td>6221/24492</td>
<td>986</td>
<td>2538</td>
<td>163</td>
</tr>
</tbody>
</table>

Each of the tables contains three rows corresponding to: (1) regular coarser mesh; (2) mesh with local refinement in subdomains (hatched in the figures) of some singularities of the solution; (3) globally refined mesh (2) with a given reduction factor \( R_f = 4 \) in the first two test problems, and \( R_f = 6 \) for the last one) with respect of the maximal area of the triangles. Homogeneous Dirichlet boundary conditions are assumed along the the \( G_D \) parts of boundaries.
BENCHMARK 1. A rectangular concrete console block of a size $5 \times 8$ m with an artificial crack is stretched along its $8$ m sides (see Fig. (1)). The material is homogeneous with $E = 2.1 \times 10^9$ Pa and $\nu = 0.17$. Local refinement around the crack is applied to get the second mesh. The results presented in Table 2 very well illustrate the convergence properties of the proposed DD preconditioners in a relatively simple local refinement case.

![Figure 2](image)

**Fig. 2. Benchmark 2. Two-layer slope problem.**

### Table 3

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Number of nodes</th>
<th>CG</th>
<th>SDC MIC(0)</th>
<th>ISO MIC(0)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear/Quadratic</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1594/6224</td>
<td>1077</td>
<td>57</td>
<td>95</td>
</tr>
<tr>
<td>2</td>
<td>2186/8571</td>
<td>1491</td>
<td>62</td>
<td>107</td>
</tr>
<tr>
<td>3</td>
<td>6248/24826</td>
<td>2282</td>
<td>84</td>
<td>149</td>
</tr>
</tbody>
</table>

BENCHMARK 2. A two-layer soil body illustrating a slope stability computer simulation is considered (see Fig. 2) where the material characteristics are as follows: sandy clay layer $L_1$ with $E_{L_1} = 8 \times 10^6$ Pa and $\nu_{L_1} = 0.35$; and clay layer $L_2$ with $E_{L_2} = 22 \times 10^6$ Pa and $\nu_{L_2} = 0.18$. The first layer is loaded by the absorbed water (35% of the layer volume). There is also an external vertical load modelling a railway passing over the slope. The local refinement is applied in a strip along the interface boundary between the different layers, and under the seal of the external load. This problem is somehow more complicated as a moderate jump of the coefficients appears. From other point of view the solution is more regular, and as we see in Table 3, the convergence of the iterative methods is even faster then for the first benchmark. Note that the number of the nodes of the finest meshes are almost the same for the first two benchmark problems.

BENCHMARK 3. The interaction between a vertical concrete wall and a weak multilayer soil media is studied (see Fig. 3). The characteristics of the problem are as follows: concrete wall with coefficients $E_C = 3.45 \times 10^9$ Pa and $\nu_C = 0.2$; and soil layers with: $E_{L_1} = 5.2 \times 10^6$ Pa and $\nu_{L_1} = 0.4$; $E_{L_2} = 9.4 \times 10^6$ Pa and $\nu_{L_2} = 0.35$; $E_{L_3} = 14.0 \times 10^6$ Pa and $\nu_{L_3} = 0.25$; $E_{L_4} = 21.4 \times 10^6$ Pa and $\nu_{L_4} = 0.2$. The scheme of the loading is presented in the figure where the external forces are assumed uni-
formly distributed across the top side of the wall. The zone of local refinement covers the wall.

\[ F = -52083 \text{ N/m} \]
\[ F = 43403 \text{ N/m} \]

![Diagram of wall with labels](image)

**Fig. 3. Benchmark 3. Foundation wall in multilayer soil media.**

> From computational point of view, this problem is the hardest one, due to the strong jump of the elasticity modules between the concrete and the surrounding weak soil. There is also a strongly expressed zone with stresses concentration under the bottom of the wall. The increased number of iterations presented in Table 4 well illustrate the influence of these factors on the convergence rate of the iterative methods.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Number of nodes</th>
<th><strong>CG</strong></th>
<th><strong>SDC MIC(0)</strong></th>
<th><strong>ISO MIC(0)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear/Quadratic 1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1574/6041</td>
<td>5697</td>
<td>26118</td>
<td>269</td>
</tr>
<tr>
<td>2</td>
<td>3115/13481</td>
<td>16127</td>
<td>59260</td>
<td>470</td>
</tr>
<tr>
<td>3</td>
<td>9195/36586</td>
<td>20616</td>
<td>95952</td>
<td>1335</td>
</tr>
</tbody>
</table>

**Table 4**


---

5. **Concluding remarks.** Two displacement decomposition preconditioning techniques for PCG iterative solution of 2D elasticity problems are considered in this paper where linear and quadratic Lagrangian finite elements are applied for discretization of the Navier system of partial differential equations. The proposed approach is implemented in the developed general purpose FEM package. The modified incomplete Cholesky factorization MIC(0) is used to approximate the obtained diagonal blocks where the stability of the factorization is controlled by the parameters of the incorporated mesh generator TRIANGLE. The robustness of the studied algorithms for non-structured meshes, including local refinement in a very general setting, is confirmed by the presented numerical tests. We would like especially to stress the attention of the reader on the efficiency of the algorithms when quadratic finite elements are used. Another important conclusion is that the ISO MIC(0) preconditioner demonstrated some advantages for the real-life problems which is particularly strongly expressed for the most complicated last benchmark problem. We have also
observed that the number of the ISO MIC(0) iterations have a behaviour very near to $O(N^{9/4})$ for all of the test problems.

Our final conclusion is that the positive experience with the developed 2D FEM package is a good starting point for the next step research devoted to the more complicated, but definitely more important, 3D case.

REFERENCES