P-Finsler spaces with vanishing Douglas tensor

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Abstract. The purpose of the present paper is to prove that a *P-Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater then three.

1. Introduction

Let $F^n (M^n, L)$ be an $n$-dimensional Finsler space, where $M^n$ is a connected differentiable manifold of dimension $n$ and $L(x, y)$ is the fundamental function defined on the manifold $T(M) \setminus 0$ of nonzero tangent vectors. Let us consider a geodesic curve $x^i = x^i(t)$, $t_0 \leq t \leq t_1$. The system of differential equations for geodesic curves of $F^n$ with respect to canonical parameter $t$ is given by

$$\frac{d^2 x^i}{dt^2} = -2G^i(x, y), \quad y^i = \frac{dx^i}{dt},$$

where

$$G^i = \frac{1}{4} g^{ir} \left( y^s \left( \frac{\partial L^2}{\partial x^s} \right) - \frac{\partial L^2}{\partial x^r} \right),$$

$$g_{ij} = \frac{1}{2} L^2_{(i)(j)}, \quad (i) = \frac{\partial}{\partial y^i}, \text{ and } (g^{ij}) = (g_{ij})^{-1}.$$

The Berwald connection coefficients $G^i_j(x, y), G^i_{jk}(x, y)$ can be derived from the function $G^i$, namely $G^i_j = G^i_{(j)}$ and $G^i_{jk} = G^i_{(jk)}$. The Berwald covariant derivative with respect to the Berwald connection can be written as

$$T^i_{j;k} = \partial T^i_j/\partial x^k - T^i_{(j)r} G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

(Throughout the present paper we shall use the terminology and definitions described in Matsumoto’s monograph [6].)

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1 The Roman indices run over the range $1, \ldots, n$. 
2. Douglas tensor, Randers metric, *P-space

Let us consider two Finsler space $F^n(M^n, L)$ and $\bar{F}^n(M^n, \bar{L})$ on a common underlying manifold $M^n$. A diffeomorphism $F^n \rightarrow \bar{F}^n$ is called geodesic if it maps an arbitrary geodesic of $F^n$ to a geodesic of $\bar{F}^n$. In this case the change $L \rightarrow \bar{L}$ of the metric is called projective. It is well-known that the mapping $F^n \rightarrow \bar{F}^n$ is geodesic iff there exist a scalar field $p(x, y)$ satisfying the following equation

\begin{equation}
\tilde{G}^i = G^i + p(x, y)y^i, \quad p \neq 0.
\end{equation}

The projective factor $p(x, y)$ is a positive homogeneous function of degree one in $y$. From (2) we obtain the following equations

\begin{equation}
\tilde{G}^i_j = G^i_j + p_j \delta^i_j + p_j y^i, \quad p_j = p(j),
\end{equation}

\begin{equation}
\tilde{G}^i_{jk} = G^i_{jk} + p_j \delta^i_k + p_k \delta^i_j + p_j y^i, \quad p_{jk} = p_{j(k)},
\end{equation}

\begin{equation}
\tilde{G}^i_{jkl} = G^i_{jkl} + p_{jk} \delta^i_l + p_{jl} \delta^i_k + p_{kl} \delta^i_j + p_{jkl} y^i, \quad p_{jkl} = p_{j(k)l}.
\end{equation}

Substituting $p_{ij} = (\tilde{G}^i_j - G^i_j) / (n+1)$ and $p_{ijk} = (\tilde{G}^i_{jk} - G^i_{jk}) / (n+1)$ into (5) we obtain the so called Douglas tensor which is invariant under geodesic mappings, that is

\begin{equation}
D^i_{jkl} = G^i_{jkl} - (y^i G_{j(k)} + \delta^i_j G_{kl} + \delta^i_k G_{jl} + \delta^i_l G_{jk}) / (n+1),
\end{equation}

which is invariant under geodesic mappings, that is

\begin{equation}
D^i_{jkl} = \tilde{D}^i_{jkl}.
\end{equation}

We now consider some notions and theorems for special Finsler spaces.

**Definition 1.** ([1]) In an $n$-dimensional differentiable manifold $M^n$ a Finsler metric $L(x, y) = \alpha(x, y) + \beta(x, y)$ is called Randers metric, where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric in $M^n$ and $\beta(x, y) = b_i(x)y^i$ is a differential 1-form in $M^n$. The Finsler space $F^n = (M^n, L) = \alpha + \beta$ with Randers metric is called Randers space.

**Definition 2.** ([1]) The Finsler metric $L = \alpha^2 / \beta$ is called Kropina metric. The Finsler space $F^n = (M^n, L) = \alpha^2 / \beta$ with Kropina metric is called Kropina space.
Definition 3. ([1], [6]) A Finsler space of dimension \( n > 2 \) is called \( C \)-reducible, if the tensor \( C_{ijk} = \frac{1}{2}g_{ij(k)} \) can be written in the form

\[
C_{ijk} = \frac{1}{n+1} \left( h_{ij}C_k + h_{ik}C_j + h_{jk}C_i \right),
\]

where \( h_{ij} = g_{ij} - l_il_j \) is the angular metric tensor and \( l_i = L(i) \).

Theorem 1. ([7]) A Finsler space \( F^n, n \geq 3 \), is \( C \)-reducible iff the metric is a Randers metric or a Kropina metric.

Definition 4. ([4], [5]) A Finsler space \( F^n \) is called *P-Finsler space, if the tensor \( P_{ijk} = \frac{1}{2}g_{ij(k)} \) can be written in the form

\[
P_{ijk} = \lambda(x, y)C_{ijk}.
\]

Theorem 2. ([4]) For \( n > 3 \) in a \( C \)-reducible *P-Finsler space \( \lambda(x, y) = k(x)L(x, y) \) holds and \( k(x) \) is only the function of position.

3. *P-Randers space with vanishing Douglas tensor

Definition 5. ([3]) A Finsler space is said to be of Douglas type or Douglas space, iff the functions \( G^i y^j - G^j y^i \) are homogeneous polynomials in \((y^i)\) of degree three.

Theorem 3. ([3]) A Finsler space is of Douglas type iff the Douglas tensor vanishes identically.

Theorem 4. ([5]) For \( n > 3 \), in a \( C \)-reducible *P-Finsler space \( D_{jkl}^i = 0 \) holds.

If we consider a Randers change

\[
\overline{L}(x, y) \rightarrow L(x, y) + \beta(x, y),
\]

where \( \beta(x, y) \) is a closed one-form, then this change \( \overline{L} \rightarrow L \) is projective.

Definition 6. ([1]) A Finsler space is called Landsberg space if the condition \( P_{ijk} = 0 \) holds.

Theorem 5. ([2]) If there exist a Randers change with respect to a projective scalar \( p(x, y) \) between a Landsberg and a *P-Finsler space (fulfilling the condition \( \overline{P}_{ijk} = p(x, y)\overline{C}_{ijk} \)), then \( p(x, y) \) can be given by the equation

\[
p(x, y) = e^{\varphi(x)}\overline{L}(x, y).
\]
It is well-known that the Riemannian space is a special case of the Landsberg space. In a Riemannian space we have $D_{ijkl} = 0$, and a *P-Randers space with a closed one-form $\beta(x, y)$ is a Finsler space with vanishing Douglas tensor. 

**Theorem 6.** ([3]) A Randers space is a Douglas space iff $\beta(x, y)$ is a closed form. Then 

\[
2G^i = \gamma^i_{jk} y^j y^k + \frac{r_{lm} y^l y^m}{\alpha + \beta} y^i,
\]

where $\gamma^i_{jk}(x)$ is the Levi–Civita connection of a Riemannian space, $r_{lm}$ is equal to $b_{i;ij}$ hence $r_{lm}$ depends only on position.

From the Theorem 6. and (10) follows that 

\[
\frac{r_{lm} y^l y^m}{\alpha + \beta} = e^{\varphi(x)}(\alpha + \beta)
\]

that is 

\[
\frac{r_{lm} y^l y^m}{\mathcal{L}} = e^{\varphi(x)} \mathcal{L}.
\]

From the last equation we obtain 

\[
r_{lm} y^l y^m = e^{\varphi(x)} \mathcal{L}^2.
\]

Differentiating twice this equation by $y^l$ and $y^m$ we get 

\[
b_{i;ij} = e^{\varphi(x)} g_{ij}.
\]

This means that the metrical tensor $g_{ij}$ depends only on $x$, so we get the following 

**Theorem.** A *P-Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater than three.

### 4. Further possibilities

From Theorem 1, Theorem 4 and our Theorem follows that only the *P-Kropina spaces can be *P-C reducible spaces with vanishing Douglas tensor which are different from Riemannian spaces. We would like to investigate this letter case in a forthcoming paper.
References


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