Eigenvalue Criteria For Existence And Nonexistence Of Bounded And Unbounded Positive Solutions To A Third-Order BVP On The Half Line*

Abdelhamid Benmezaï†, Salima Mechrouk‡, El-Djouher Sedkaoui§

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Abstract

Under eigenvalue criteria, we establish in this article existence and nonexistence results for positive solutions to the third-order boundary value problem

\[
\begin{align*}
-u'''(t) + k^2 u'(t) &= f(t, u(t)), \quad t > 0 \\
u(0) &= u'(0) = u'(+\infty) = 0,
\end{align*}
\]

where \( k \) is a positive constant and the function \( f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \) is continuous. The boundedness and the unboundedness of the solution are also discussed.

1 Introduction and Main Results

Because third order ordinary differential equations arise in modeling various physical phenomena, the study of existence of solutions to boundary value problems (bvp for short) related to these, is a rapidly growing branch of applied mathematics. As examples, we start by Danziger and Elemergreen who proposed in [15] (see p. 133) the following third-order linear differential equations

\[
\begin{align*}
3y''' + 2y'' + 1y' + (1 + k)y &= kc, \quad <c, \\
3y''' + 2y'' + 1y' &= 0, \quad >c, \quad (1)
\end{align*}
\]

to describe the variation of thyroid hormone with time. Notice that the unknown \( y = y(t) \) in Equation (1) represents the concentration of thyroid hormone at time \( t \) and \( \alpha_3, \alpha_2, k \) and \( c \) are constants.

Motivated by the asymptotic behavior of the solutions of Volterra integro-differential equations having the form

\[
\begin{align*}
y'(t) &= \gamma y(t) + \int_0^t (\lambda + \mu t + \vartheta s) y(s) ds, \quad t \geq 0, \\
y(0) &= 1,
\end{align*}
\]

Jackiewicz et al. have investigated in [20] the third-order differential equations of the type

\[
u''' = \gamma u'' + (\lambda + (\mu + \vartheta) t) u' + (2\mu + \vartheta) u, \quad (2)
\]

where \( \lambda, \gamma, \mu \) and \( \vartheta \) are real parameters and \( \mu + \vartheta = 0 \).

As a simple model exhibiting many of the features of the Hodgkin–Huxley equations, Nagumo proposed (see [27]) third-order differential equation

\[
y''' - cy'' + f'(y)y' - \frac{b}{c} y = 0, \quad (3)
\]

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†National High School of Mathematics, Sidi-Abdallah, Algiers, Algeria
‡Faculty of Sciences, University Mhamed Bouguera, Boumerdes, Algeria
§Faculty of Mathematics, USTHB, Algiers, Algeria
where \( f \) is a regular function.

The partial differential equation
\[
y_t + y_{xxxx} + y_{xx} + \frac{1}{2}y^2 = 0
\]
arises in a large variety of physical phenomena. Commonly known as the Kuramoto-Sivashinsky equation, it was introduced to describe pattern formation in reaction diffusion systems as well as to model the instability of flame front propagation (see Y. Kuramoto and T. Yamada [23] and D. Michelson [28]). Its traveling wave solutions (i.e. \( y(x,t) = y(x-ct) \)) are the solutions of the nonlinear third-order differential equation
\[
\theta y'''(x) + y'(x) + g(y) = 0,
\]
where the parameter \( \theta \) depends on the constant \( c \) and \( g \) is an even function.

A three-layer beam is formed by parallel layers of different materials. For an equally loaded beam of this type, Krajcinovic in [22] proved that the deflection \( u \) is governed by the third order differential equation
\[
-y''' + k^2 y' = a,
\]
where the parameters \( k \) and \( a \) depend on the elasticity of the layers.

Moreover, study of existence of positive solutions for third-order bvps has received a great deal of attention and was the subject of many articles, see, for instance, [13, 14, 16, 17, 18, 26, 30, 32, 33, 34, 35, 36], for third-order bvps posed on finite intervals and [1, 2, 3, 4, 7, 9, 10, 11, 12, 19, 21, 24, 25, 29, 31] for such bvps posed on the half-line.

In this article, we establish under eigenvalue criteria, nonexistence and existence results for positive solutions to the third-order bvp:
\[
\begin{cases}
-u'''(t) + k^2 u'(t) = f(t,u(t)), & t > 0 \\
u(0) = u'(0) = u'(+\infty) = 0,
\end{cases}
\tag{6}
\]
where \( k \in (0, +\infty) \), \( f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \) is a continuous function (\( \mathbb{R}^+ := [0, +\infty) \)) and observe that the form of the differential equation in (6) is more general to those of (1)-(5). The physical constant \( k \) will play a crucial role in building an appropriate functional framework for a fixed point formulation to the bvp (6).

In this work we mean by a positive solution to the bvp (6), a function \( u \in C^3(\mathbb{R}^+,\mathbb{R}^+) \) satisfying \( u(t_+) > 0 \) for some \( t_+ > 0 \) and all equations in the bvp (6).

When looking for positive solutions by using the fixed point theory in cones, authors often make use of the compression and expansion of a cone principle in a Banach space. This principle states that if \( P \) is a cone in a Banach space \( (B, \| \cdot \|) \), \( T : P_{r,R} \to P \) is a compact mapping where \( P_{r,R} = \{ u \in P : r \leq \| u \| \leq R \} \) and one of the following situations \( a) \) and \( b) \) holds:

a) \( \| Tu \| \geq \| u \| \) for all \( u \in P \), \( \| u \| = r \) and \( \| Tu \| \leq \| u \| \) for all \( u \in P \), \( \| u \| = R \),

b) \( \| Tu \| \leq \| u \| \) for all \( u \in P \), \( \| u \| = r \) and \( \| Tu \| \geq \| u \| \) for all \( u \in P \), \( \| u \| = R \),

then \( T \) has a fixed point \( u \) such that \( r \leq \| u \| \leq R \).

This principle has advantage to be applicable on any region of the cone \( P \) and it has the flaw that the realization of the inequality \( \| Tu \| \geq \| u \| \) requires a specific cone, see, for instance [14, 16, 26, 34, 35].

The main tool in this work consists in the fixed point theory in cones. The operator of our fixed point formulation associated to bvp (6) is defined on the Banach space of continuous functions \( u \) satisfying \( \lim_{t \to +\infty} \frac{u(t)}{t} = 0 \). Notice that this space is imposed by the boundary condition in (6) \( \lim_{t \to +\infty} u'(t) = 0 \), since by the L’Hospital’s rule \( \lim_{t \to -\infty} \frac{u(t)}{t} = \lim_{t \to +\infty} u'(t) = 0 \). Unfortunately, the cone of nonnegative function lying in the above space does not offer the possibility to realize the inequality \( \| Tu \| \geq \| u \| \). To overcome this difficulty we use the approach exposed in Section 3. This approach gives a necessary condition for existence of positive solution (see Proposition 3), and has the advantage to be applicable in any cone. However, it has the disadvantage that the radii \( r \) and \( R \) must be taken near 0 and \( +\infty \) respectively. In other
words we lose the localization established in the compression and expansion of a cone principal in a Banach space, \( r \leq \|w\| \leq R \).

Since a function \( u \) satisfying \( \lim_{t \to +\infty} \frac{u(t)}{t} = 0 \) may be bounded or unbounded (e.g. \( u(t) = \ln(1 + t) \)), we provide in each existence result established in this paper sufficient conditions for the boundedness or unboundedness of the obtained positive solution. In this paper, we let

\[
\Gamma = \left\{ q \in C(\mathbb{R}^+, \mathbb{R}^+) : q(s) > 0 \ \text{a.e.} \ s > 0 \right\},
\]

\[
\Gamma_0 = \left\{ q \in \Gamma : \sup_{s \geq 0} q(s) < \infty \right\},
\]

\[
\Gamma_1 = \left\{ q \in \Gamma : \lim_{s \to +\infty} q(s) = 0 \right\},
\]

\[
\Gamma_2 = \left\{ q \in \Gamma : \lim_{s \to +\infty} q(s) = 0 \text{ and } \int_0^{+\infty} q(s)ds < \infty \right\},
\]

\[
\Delta_i = \{ q \in \Gamma : qp_i \in \Gamma_i \} \text{ for } i = 0, 1, 2,
\]

\[
\Delta_3 = \{ q \in \Gamma : qp_3 \in \Gamma_1 \},
\]

\[
\Delta = \Delta_1 \cup \Delta_2,
\]

where

\[
p_1(t) = 1 + t, \quad p_0(t) = p_2(t) = 1, \quad p_3(t) = e^{kt}.
\]

Notice that \( \Gamma_2 \subset \Gamma_1 \subset \Gamma_0, \Delta_2 = \Gamma_2, \Delta_3 \subset \Delta_1 \cap \Delta_2, \Delta_1 \setminus \Delta_2 \neq \emptyset \) and \( \Delta_2 \setminus \Delta_1 \neq \emptyset \). Indeed, for

\[
q_1(s) = \frac{1}{(1 + s) \ln(4 + s)}, \quad q_2(s) = \frac{m(s)}{1 + s},
\]

where

\[
m(s) = \begin{cases} 
2n^4s - n(2n^4 - 1) & \text{if } s \in \left[n - \frac{1}{2n^3}, n\right], \\
-2n^4s + n(2n^4 + 1) & \text{if } s \in \left[n, n + \frac{1}{2n^3}\right], \\
0 & \text{otherwise},
\end{cases}
\]

we have \( q_1 \in \Delta_1 \setminus \Delta_2 \) and \( q_2 \in \Delta_2 \setminus \Delta_1 \).

A continuous mapping \( g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) is said to be

- a \( \Gamma_r \)-Caratheodory function for \( i = 0, 1, 2 \), if for all \( r > 0 \) there exists a function \( \psi_r \in \Gamma_i \) such that

\[
|g(t, p_i(t)u)| \leq \psi_r(t) \text{ for all } t \geq 0 \text{ and } u \in [-r, r].
\]

- a \( \Gamma_{2+r} \)-Caratheodory function for \( i = 1, 2 \), if for all \( r > 0 \) there exists a function \( \psi_r \in \Gamma_i \) such that

\[
|g(t, p_i(t)u)| \leq \psi_r(t) \text{ for all } t \geq 0 \text{ and } u \in [-r, r].
\]

Consider for \( q \in \Delta \), the linear eigenvalue problem associated with the bvp (6)

\[
\begin{cases}
-u''(t) + k^2u'(t) = \mu q(t)u(t), & t > 0 \\
u(0) = u'(0) = u'(+\infty) = 0,
\end{cases}
\]

\[
(7)
\]

where \( \mu \) is a real parameter.

A positive real number \( \mu_0 \) is said to be a positive eigenvalue of the bvp (7), if there exists a function \( \phi \in C^3(\mathbb{R}^+, \mathbb{R}^+) \) such that \( \phi(t_0) > 0 \) for some \( t_0 > 0 \) and the pair \( (\mu_0, \phi) \) satisfies all equations in the bvp (7).

The first result of this paper concerns existence of the positive eigenvalue of the bvp (7).
**Proposition 1** For all \( q \in \Delta \), the eigenvalue problem (7) admits a unique positive eigenvalue \( \mu(q) > 0 \) associated with an eigenfunction \( \phi \). Moreover, if \( q \in \Delta_2 \) then \( \phi \) is bounded and if not (i.e. \( \int_0^{+\infty} q(s)ds = +\infty \)), then \( \phi \) is unbounded, i.e. \( \lim_{t \to +\infty} \phi(t) = +\infty \).

**Theorem 1** Assume for \( i = 1 \) or 2, the nonlinearity \( f \) is a \( \Gamma_i \)-Caratheodory function and there exists a function \( q \) in \( \Delta_i \) such that either

\[
\inf \left\{ \frac{f(t, p_i(t)u)}{p_i(t)q(t)u} : t, u > 0 \right\} > \mu(q) \tag{8}
\]

or

\[
\sup \left\{ \frac{f(t, p_i(t)u)}{p_i(t)q(t)u} : t, u > 0 \right\} < \mu(q). \tag{9}
\]

Then the bvp (6) admits a positive solution.

The statements of the following existence results need additional notations. For any \( \Gamma_i \)-Caratheodory function \( g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) and \( q \in \Delta_i \) with \( i \in \{0, 1, 2, 3\} \) and \( \nu = 0, +\infty \), we set

\[
g_{i,\nu}^+(q) = \limsup_{u \to \nu} \left( \max_{t \geq 0} \frac{g(t, p_i(t)u)}{p_i(t)q(t)u} \right)
\]

and

\[
g_{i,\nu}^-(q) = \liminf_{u \to \nu} \left( \min_{t \geq 0} \frac{g(t, p_i(t)u)}{p_i(t)q(t)u} \right).
\]

**Theorem 2** Suppose for \( i = 1 \) or 2, the function \( f \) is \( \Gamma_i \)-Caratheodory and there are two functions \( q_0 \) and \( q_\infty \) in \( \Delta_i \) such that either

\[
\frac{\int_{i,\infty}^+(q_\infty)}{\mu(q_\infty)} < 1 < \frac{\int_{i,0}^-(q_0)}{\mu(q_0)} \leq \frac{\int_{i,0}^+(q_0)}{\mu(q_0)} < \infty \tag{10}
\]

or

\[
\frac{\int_{i,0}^+(q_0)}{\mu(q_0)} < 1 < \frac{\int_{i,\infty}^+(q_\infty)}{\mu(q_\infty)} \leq \frac{\int_{i,\infty}^-(q_\infty)}{\mu(q_\infty)} < \infty. \tag{11}
\]

Then the bvp (6) admits a solution \( u \) in \( K_i \). Moreover, if \( i = 2 \) then \( u \) is bounded and if \( i = 1 \) and

\[
\lim_{t \to +\infty} \int_1^t f(s, p_1(s)\lambda)ds = +\infty \text{ uniformly for } \lambda \text{ in compact intervals of } (0, +\infty),
\]

then \( u \) is unbounded.

In Theorem 2, conditions (10) and (11) impose the nonlinearity \( f \) to be sublinear at \( +\infty \), that is there is a positive constants \( d \) and a function \( c \in \Gamma_i \) such that \( f(t, u) \leq c(t)u \) for all \( u \geq d \) and \( t \geq 0 \). To avoid such a condition, we have been led to look for positive solutions in the largest Banach space. We have obtained then the following result.

**Theorem 3** Suppose that the function \( f \) is \( \Gamma_3 \)-Caratheodory and there are two functions \( q_0 \) and \( q_\infty \) in \( \Delta_3 \) such that either

\[
\frac{\int_{3,\infty}^+(q_\infty)}{\mu(q_\infty)} < 1 < \frac{\int_{3,0}^-(q_0)}{\mu(q_0)}, \tag{13}
\]

or

\[
\frac{\int_{3,0}^+(q_0)}{\mu(q_0)} < 1 < \frac{\int_{3,\infty}^-(q_\infty)}{\mu(q_\infty)}. \tag{14}
\]

Then the bvp (6) admits a positive solution \( u \). Moreover, if the nonlinearity \( f \) is a \( \Gamma_4 \)-Caratheodory function then the solution \( u \) is bounded, and if

\[
\lim_{t \to +\infty} \int_1^t f(s, p_3(s)\lambda)ds = +\infty \text{ uniformly for } \lambda \text{ in compact intervals of } (0, +\infty),
\]

then \( u \) is unbounded.
Consider now, the particular version of the bvp (6) where the nonlinearity $f$ takes the form $f(t,u) = q_*(t)h(t,u)$; namely, we consider the bvp

$$
\begin{cases}
-u'''(t) + k^2u'(t) = q_*(t)h(t,u(t)), & t > 0, \\
u(0) = u'(0) = u'(+\infty) = 0,
\end{cases}

(16)
$$

where $q_* \in \Gamma$ and $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function.

If $h/p_i$ is a $\Gamma_0$-Caratheodory function for $i = 1, 2$ or $3$, then we set for $\nu = 0, +\infty$,

$$h^+_{i,\nu} = h^+_{i,\nu}(1), \quad h^-_{i,\nu} = h^-_{i,\nu}(1).$$

We obtain respectively from Theorems 1, 2 and 3 the following corollaries:

**Corollary 1** Assume for $i = 1$ or $2$ that $q_* \in \Delta_i$, the function $h/p_i$ is $\Gamma_0$-Caratheodory and either

$$\inf \left\{ \frac{h(t,p_i(t)u)}{p_i(t)u} : t, u > 0 \right\} > \mu(q),$$

or

$$\sup \left\{ \frac{f(t,p_i(t)u)}{p_i(t)u} : t, u > 0 \right\} < \mu(q).$$

Then the bvp (16) has no positive solution.

**Corollary 2** Assume for $i = 1$ or $2$ that $q_* \in \Delta_i$, the function $h/p_i$ is $\Gamma_0$-Caratheodory and either

$$h^+_{i,\infty} < \mu(q_*) < h^-_{i,0} \leq h^+_{i,0} < \infty,$$

or

$$h^+_{i,0} < \mu(q_*) < h^-_{i,\infty} \leq h^+_{i,\infty} < \infty.$$

Then the bvp (16) admits a positive solution. Moreover, if $i = 2$ then $u$ is bounded and if $i = 1$ and

$$\lim_{t \to +\infty} \int_1^t q_*(s)h(s,p_1(s)\lambda)ds = +\infty \text{ uniformly for } \lambda \text{ in compact intervals of } (0, +\infty),$$

then $u$ is unbounded.

**Corollary 3** Suppose that $q_* \in \Delta_3$, the function $h/p_3$ is $\Gamma_0$-Caratheodory and either

$$h^+_{3,\infty} < \mu(q_*) < h^-_{3,0},$$

or

$$h^+_{3,0} < \mu(q_*) < h^-_{3,\infty}.$$

Then the bvp (16) admits a positive solution. Moreover, if $q_* \in \Delta_2$ then $u$ is bounded and if

$$\lim_{t \to +\infty} \int_1^t q_*(s)h(s,p_3(s)\lambda)ds = +\infty \text{ uniformly for } \lambda \text{ in compact intervals of } (0, +\infty),$$

then $u$ is unbounded.
2 Example

Consider for $i = 1, 2, 3$ the bvp (6) with

$$f(t, u) = F_i(t, u) = Aq_0(t) - \frac{p_i(t)u}{(p_i(t))^2 + u^2} + Bq_\infty(t) \frac{u^2}{p_i(t) + u},$$

where $A$ and $B$ are positive real numbers and $q_0, q_\infty \in \Delta_i$.

It is easy to see that $F_i$ is a $\Gamma_i$-Caratheodory function and if

$$0 < \inf_{t \geq 0} \frac{q_\infty(t)}{q_0(t)} \leq \sup_{t \geq 0} \frac{q_\infty(t)}{q_0(t)} < \infty,$$

then

$$f_{i,0}^-(q_0) = f_{i,0}^+(q_0) = A$$

and

$$f_{i,\infty}^-(q_\infty) = f_{i,\infty}^+(q_\infty) = B.$$

We deduce from Theorems 2 and 3 that for such a nonlinearity $f$, the bvp (6) admits a solution if either

$$A < \mu(q_0) \text{ and } B > \mu(q_\infty)$$

or

$$A > \mu(q_0) \text{ and } B < \mu(q_\infty).$$

Evidently for $i = 2$, the obtained solution $u$ is bounded and for $i = 1$, if $\int_0^{+\infty} q_0 p_1 ds = +\infty$ then $u$ is unbounded. Indeed, for any interval $[a, b] \subset (0, +\infty)$ we have

$$\int_1^t f(s, p_2(s)\lambda)ds \geq A \int_1^t q_0(s)p_1(s)\frac{\lambda}{1 + \lambda^2}ds$$

$$\geq Aa \int_1^t q_0(s)p_1(s)ds \to +\infty \text{ as } t \to +\infty.$$

For instance if $q_0(t) = q_\infty(t) = (1 + t)^{-2}$ the obtained solution is unbounded.

In the case $i = 3$, if $\int_1^{+\infty} q_0(s)p_3(s)ds < +\infty$ then the solution is bounded and if $\int_1^{+\infty} q_0(s)p_3(s)ds = +\infty$, the same computations as above lead us to $u$ is unbounded. For example, if $q_0(t) = q_\infty(t) = (1 + t)^{-1} e^{-kt}$, then the obtained solution is unbounded.

3 Abstract Background

In this section we let $(Z, \|\|)$ be a real Banach space and by $L(Z)$ and $r(L)$ we refer respectively to the set of all linear bounded self-mapping defined on $Z$ and the spectral radius of an operator $L$ in $L(Z)$. We let also $C$ be a cone in $Z$, that is $C$ is a nonempty closed convex subset of $Z$ such that $C \cap (-C) = \{0_Z\}$ and $tC \subset C$ for all $t \geq 0$. In the reminder of this section, the notation $\preceq$ refers to the partial order induced by the cone $C$ on the Banach space $Z$. We write for all $u, v \in Z$: $u \preceq v$ (or $v \geq u$) if $v - u \in C$ and $u \prec v$ (or $v \succ u$) if $v - u \in C \setminus \{0_Z\}$.

**Definition 1** A compact operator $L$ in $L(Z)$ is said to be

i) positive, if $L(C) \subset C$,

ii) strongly positive, if $\text{int}(C) \neq \emptyset$ and $L(C \setminus \{0_Z\}) \subset \text{int}(C)$,

iii) lower bounded on the cone $C$, if

$$\inf \{\|Lu\| : u \in C \cap \partial B(0_Z, 1)\} > 0.$$
Hereafter we denote by $\mathcal{L}_C(Z)$ the subset of all positive compact operators in $\mathcal{L}(Z)$ and for any operator $L$ in $\mathcal{L}_C(Z)$ we define the sets:

\[
\Lambda_L = \{ \theta \geq 0 : \exists u > 0_Z \text{ such that } Lu \geq \theta u \} \quad \text{and} \quad \Gamma_L = \{ \theta \geq 0 : \exists u > 0_Z \text{ such that } Lu \leq \theta u \}.
\]

It is proved in [5] that for all $L$ in $\mathcal{L}_C(Z)$

\[
\sup L \geq \inf \Gamma_L.
\]

**Definition 2** An operator $L$ in $\mathcal{L}_C(Z)$ is said to have the strongly index-jump property (SIJP for short) at $\mu$, where $\mu$ is a positive real number, if

\[
\mu = \sup L = \inf \Gamma_L.
\]

**Proposition 2 (Proposition 3.16 in [5])** Suppose that $L$ is an operator in $\mathcal{L}_C(Z)$. If $L$ is strongly positive then $L$ has the SIJP at $r(L)$.

**Theorem 4 (Theorem 3.23 in [5])** Assume that $L \in \mathcal{L}_C(Z)$ and $(L_n) \subset \mathcal{L}_C(Z)$ are such that $(L_n)$ is increasing, for all integers $n \geq 1$, $L_n$ has the SIJP at $\mu_n$ and $L_n \rightarrow L$ in operator norm. Then $L$ has the SIJP at $\mu = \lim \mu_n = \sup \mu_n$.

**Remark 1** From Proposition 3.14 and Proposition 3.15 in [6] we conclude that if $L \in \mathcal{L}_C(Z)$ has the SIJP at $\mu$ then $\mu$ is the unique positive eigenvalue of $L$.

**Remark 2** Observe that if $L \in \mathcal{L}_C(Z)$ has the SIJP at $\mu$ and $L(C) \subset P \subset C$ where $P$ is a cone in $Z$, then $L \in \mathcal{L}_P(Z)$ has the SIJP at $\mu$.

Our approach in this work is based on a fixed point formulation of the bvp (6). More exactly, we will show that the problem of existence and nonexistence of positive solutions to the bvp (6) is equivalent to that of existence and nonexistence of fixed point for a completely continuous mapping defined on some cone in an appropriate functional space. The following proposition and theorems will be used to prove the main results of this paper.

Let $T : C \rightarrow C$ be a completely continuous mapping. We start by the proposition below which provide under an eigenvalue criteria a nonexistence result of fixed point to the mapping $T$.

**Proposition 3** Suppose that there is an operator $L$ in $\mathcal{L}_C(Z)$ having the SIJP at $\mu$ such that either

\[
\mu > 1 \quad \text{and} \quad Tu \geq Lu \quad \text{for all } u \in C,
\]

or

\[
\mu < 1 \quad \text{and} \quad Tu \leq Lu \quad \text{for all } u \in C
\]

holds. Then $T$ has no fixed point.

**Proof.** We prove the proposition in the case where (18) holds, the other case is checked in the same way. To the contrary, suppose that there is $w > 0_Z$ such that $Tw = w$. Then we have that $w = Tw \geq Lw$, that is $1 \in \Gamma_L$ and $\mu = \inf \Gamma_L \leq 1$. This contradicts the condition $\mu > 1$ of Hypothesis (18). ■

The following two theorems are respectively adapted versions of Theorem 3.24 and Theorem 3.25 in [5]. They provide solvability results to the equation $u = Tu$ under eigenvalue criteria.

**Theorem 5** Suppose that $C$ is normal and for $i = 1, 2, 3$ there exists $L_i \in \mathcal{L}_C(Z)$ and $F_i : C \rightarrow C$ such that

\[
\begin{cases}
L_2 \text{ has the SIJP at } r(L_2), \\
0 < r(L_2) < 1 < r(L_1) \quad \text{and} \\
Tv \leq L_1 v + F_1 v, \\
L_2 v - F_2 v \leq Tv \leq L_3 v + F_3 v \quad \text{for all } v \in C.
\end{cases}
\]
If either
\[ F_1 v = o \left( \| v \| \right) \text{ as } v \to 0 \text{ and } F_i v = o \left( \| v \| \right) \text{ as } v \to \infty \text{ for } i = 2, 3 \] (20)
or
\[ F_1 v = o \left( \| v \| \right) \text{ as } v \to \infty \text{ and } F_i v = o \left( \| v \| \right) \text{ as } v \to 0 \text{ for } i = 2, 3, \] (21)
then \( T \) has a fixed point.

Theorem 6 Suppose that for \( i = 1, 2 \) that there is \( L_i \in \mathcal{L}_C \left( Z \right) \) and \( F_i : C \to C \) such that

\begin{align*}
&\left\{ \begin{array}{l}
L_1 \text{ has the SIJP at } r \left( L_1 \right) \\
L_1 \text{ is lower bounded on } C,
\end{array} \right.
\end{align*}
\( r \left( L_2 \right) < 1 < r \left( L_1 \right) \) and
\( L_1 v - F_1 v \leq T v \leq L_2 v + F_2 v \text{ for all } v \in C. \)

If either
\[ F_1 v = o \left( \| v \| \right) \text{ as } v \to \infty \text{ and } F_2 v = o \left( \| v \| \right) \text{ as } v \to 0 \] (22)
or
\[ F_1 v = o \left( \| v \| \right) \text{ as } v \to 0 \text{ and } F_2 v = o \left( \| v \| \right) \text{ as } v \to \infty, \] (23)
then \( T \) has a positive fixed point.

4 Fixed Point Formulation

In the reminder of this paper we let
\[ E_0 = \{ u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t \to +\infty} u(t) = 0 \}, \]
\[ E_1 = \{ u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t \to +\infty} \frac{u(t)}{1 + t} = 0 \}, \]
\[ E_2 = \{ u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t \to +\infty} u(t) = l \in \mathbb{R} \}, \]
\[ E_3 = \{ u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t \to +\infty} e^{-kt} u(t) = 0 \}. \]

Endowed respectively with the norms
\[ \| u \|_1 = \sup_{t \geq 0} \frac{|u(t)|}{1 + t}, \quad \| u \|_2 = \sup_{t \geq 0} |u(t)| \quad \text{and} \quad \| u \|_3 = \sup_{t \geq 0} e^{-kt} |u(t)|, \]
\( E_1, E_2 \) and \( E_3 \) become Banach spaces.

We let also, \( K_1, K_2 \) and \( K_3 \) be respectively the cones in \( E_1, E_2 \) and \( E_3 \) defined by
\[ K_1 = \{ u \in E_1 : u(t) \geq 0 \text{ for all } t \geq 0 \text{ and } u \text{ is nondecreasing} \}, \]
\[ K_2 = \{ u \in E_2 : u(t) \geq 0 \text{ for all } t \geq 0 \}, \]
\[ K_3 = \{ u \in E_3 : u(t) \geq \gamma(t) \| u \|_3 \text{ for all } t \geq 0 \}, \]
where
\[ \gamma(t) = \frac{1}{3k} \left( e^{-3kt} - 3e^{-kt} + 2 \right). \]

Let \( G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) be the function given by
\[ G(t, s) = \frac{1}{k^2} \left\{ \begin{array}{ll}
e^{-ks} (\cosh(kt) - 1), & \text{if } t \leq s, \\
e^{-kt} \sinh(ks) + (1 - e^{-ks}), & \text{if } s \leq t.
\end{array} \right. \]
The functions $G$ and $\frac{\partial G}{\partial t}$ are continuous and they have the following properties:

\[ G(t, s) > 0 \text{ for all } t, s > 0, \]  
\[ \frac{\partial G}{\partial t}(t, s) > 0 \text{ for all } t, s > 0, \]  
\[ G(0, s) = \frac{\partial G}{\partial t}(0, s) = 0 \text{ for all } s \in \mathbb{R}^+, \]  
\[ \lim_{t \to +\infty} G(t, s) = \frac{1}{k^2} (1 - e^{-ks}) \text{ for all } s \in \mathbb{R}^+, \]  
\[ \int_0^{+\infty} G(t, s) ds = \frac{1}{k^2} t - \frac{1}{k^3} (1 - e^{-kt}) \text{ for all } t \geq 0, \]  
\[ \sup_{t \geq 0} \frac{1}{1+t} \int_0^{+\infty} G(t, s) ds = \frac{1}{k^2}, \]  
\[ \int_0^{+\infty} |G(t_2, s) - G(t_1, s)| ds \leq \frac{2}{k^2} |t_2 - t_1| \text{ for all } t_2, t_1 \geq 0. \]  

Properties (24)–(28) and (29) are obvious and Property (30) is obtained from Property (28) for each of the cases $t_2 \geq t_1$ and $t_2 \leq t_1$.

**Lemma 1** For all functions $v$ in $E_0$, $u(t) = \int_0^{+\infty} G(t, s)v(s)ds$ is the unique solution of the bvp

\[
\begin{aligned}
-w'''(t) + k^2 w' &= v, \quad \text{in } (0, +\infty), \\
w(0) = w'(0) = w'(t) &= 0.
\end{aligned}
\]  

Moreover $u$ belongs to $E_1$.

**Proof.** Let $v \in E_0$. For any $t \geq 0$ we have by Property (28),

\[ |u(t)| = \left| \int_0^{+\infty} G(t, s)v(s)ds \right| \leq \|v\|_2 \int_0^{+\infty} G(t, s) ds < \infty. \]

Furthermore, for any $t_1, t_2 \geq 0$, we have by Property (30),

\[ |u(t_2) - u(t_1)| = \left| \int_0^{+\infty} G(t_2, s)v(s)ds - \int_0^{+\infty} G(t_1, s)v(s)ds \right| \]
\[ \leq \int_0^{+\infty} |G(t_2, s) - G(t_1, s)| ds \|v\|_2 \]
\[ \leq \frac{2 \|v\|_2}{k^2} |t_2 - t_1|. \]

The above estimates show that $u$ is well defined and $u$ is continuous on $\mathbb{R}^+$.

Differentiating three times in the identity

\[ u(t) = -\frac{e^{-kt}}{k^2} \int_0^t \sinh(ks)v(s) ds + \frac{1}{k^2} \int_0^t (1 - e^{-ks})v(s) ds + \frac{\cosh(kt) - 1}{k^2} \int_t^{+\infty} e^{-ks} v(s) ds, \]

we find

\[ u'(t) = \frac{1}{k} \left( e^{-kt} \int_0^t \sinh(ks)v(s) ds + \sinh(kt) \int_t^{+\infty} e^{-ks} v(s) ds \right), \]
\[ u''(t) = -e^{-kt} \int_0^t \sinh(ks)v(s) ds + \cosh(kt) \int_t^{+\infty} e^{-ks} v(s) ds, \]
\[ u'''(t) = k \left( e^{-kt} \int_0^t \sinh(ks) v(s) ds + \sinh(kt) \int_t^{+\infty} e^{-ks} v(s) ds \right) - v(t) = k^2 u'(t) - v(t). \]

Hence, \( u \) satisfies \(-u'''(t) + k^2 u' = v\). Since (26) gives \( u(0) = u'(0) = 0 \), it remains to prove that \( \lim_{t \to +\infty} u'(t) = \lim_{t \to +\infty} \frac{u(t)}{1+kt} = 0 \). We have

\[ u'(t) = \int_0^{+\infty} \frac{\partial G}{\partial t}(t,s) v(s) ds = \frac{1}{k} e^{-kt} \int_0^t \sinh(ks) v(s) ds + \frac{1}{k} \sinh(kt) \int_t^{+\infty} e^{-ks} v(s) ds. \]

Using L’Hopital’s formula, we obtain

\[ \lim_{t \to +\infty} e^{-kt} \int_0^t \sinh(ks) v(s) ds = \lim_{t \to +\infty} \frac{\int_0^t \sinh(ks) v(s) ds}{e^{kt}} = \lim_{t \to +\infty} \frac{\sinh(kt)}{ke^{kt}} v(t) = 0 \]

and

\[ \lim_{t \to +\infty} \left( \sinh(kt) \int_t^{+\infty} e^{-ks} v(s) ds \right) = \lim_{t \to +\infty} \frac{\sinh(kt) \int_t^{+\infty} e^{-ks} v(s) ds}{e^{kt}} = \lim_{t \to +\infty} \frac{\int_t^{+\infty} e^{-ks} v(s) ds}{e^{kt}} = \lim_{t \to +\infty} \frac{v(t)}{k} = 0. \]

This completes the proof. ■

**Lemma 2** Assume for \( i = 1 \) or 2 the function \( g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) is a \( \Gamma_1 \)-Caratheodory. Then the operator \( T^i_g : E_i \to E_i \) where for \( u \in E_i \),

\[ T^i_g u(t) = \int_0^{+\infty} G(t,s) g(s,u(s)) ds, \]

is well defined and if \( g(t,x) \geq 0 \) for all \( t,x \geq 0 \) then \( T^i_g(K_i) \subset K_i \). Moreover, if \( u \in E_i \) is a fixed point of \( T^i_g \) then \( u \) is a solution to the bvp

\[ \begin{cases} -u'''(t) + k^2 u' = g(t,u), \text{ in } (0,+\infty), \\ u(0) = u'(0) = u''(+\infty) = 0. \end{cases} \quad (32) \]

**Proof.** Since \( \Gamma_2 \subset \Gamma_1 \), in both the cases \( i = 1 \) or 2, \( g \) is a \( \Gamma_1 \)-Caratheodory function. Hence for any \( u \in E_i \), \( g(t,u) \) belongs to \( E_0 \) and \( T^i_g u \) belongs to \( E_i \) and satisfies the bvp (31) within \( v = g(t,u) \). In the case \( i = 2 \), for \( u \in E_2 \) we have \( g(t,u) \) belongs to \( \Gamma_2 \) (i.e. \( \int_0^{+\infty} g(s,u(s)) ds < \infty \)). Therefore, Lebesgue convergence theorem and Property (27) lead to

\[ \lim_{t \to +\infty} T^2_g u(t) = \frac{1}{k^2} \int_0^{+\infty} (1 - e^{-ks}) g(s,u(s)) ds \leq \frac{1}{k^2} \int_0^{+\infty} g(s,u(s)) ds < \infty. \]

This shows that \( T^2_g \) is well defined.

At the end, we conclude by Lemma 1 that any fixed point of \( T^i_g \) in \( E_i \) is a solution to the bvp (32) and it is easy to see that if \( g \) is nonnegative then \( T^i_g(K_i) \subset K_i \) for \( i = 1,2 \). ■

**Lemma 3** Assume for \( i = 1 \) or 2 the function \( g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) is a \( \Gamma_3 \)-Caratheodory. Then the operator \( T^3_g : E_3 \to E_3 \) where for \( u \in E_3 \),

\[ T^3_g u(t) = \int_0^{+\infty} G(t,s) g(s,u(s)) ds, \]

is well defined and if \( g(t,x) \geq 0 \) for all \( t,x \geq 0 \) then \( T^3_g(K_3) \subset K_3 \). Moreover, if \( u \in E_3 \) is a fixed point of \( T^3_g \) then \( u \) is a solution to the bvp (32).
Lemma 4 Let $M$ be a nonempty subset of $E_i$, $i = 1, 2, 3$. If the following conditions hold:

(a) $M$ is bounded in $E_i$,
(b) the set \( \{ u : u(t) = \frac{x(t)}{p(t)}, \ x \in M \} \) is locally equicontinuous on \([0, +\infty)\), and

(c) the set \( \{ u : u(t) = \frac{x(t)}{p(t)}, \ x \in M \} \) is equiconvergent at \(+\infty\),

then the subset \( M \) is relatively compact in \( E_i \).

Lemma 5 Assume that the function \( g : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R} \) is \( \Gamma_1 \)-Carathéodory. Then the operator \( T^1_g \) is completely continuous.

Proof. First we prove that the operator \( T^1_g \) is continuous. To this aim let \((u_n)\) be a sequence in \( E_1 \) with \( \lim u_n = u \) in \( E_1 \), and let \( R > 0 \) and \( \psi_R \in \Gamma_2 \subset \Gamma_0 \) be such that \( \|u_n\|_1 \leq R \) for all \( n \geq 1 \) and

\[
\left| g \left( t, p_1(t) \left( \frac{u}{p_1(t)} \right) \right) \right| \leq \psi_R(t) \quad \text{for all } t \geq 0 \text{ and } u \in [-R, R].
\]

We have then

\[
\| T^1_{g,u_n} - T^1_{g,u} \|_1 = \sup_{t \geq 0} \left| \frac{T^1_{g,u_n}(t) - T^1_{g,u}(t)}{p_1(t)} \right| \leq \sup_{t \geq 0} \Phi_n(t),
\]

where

\[
\Phi_n(t) = \frac{1}{p_1(t)} \int_0^{+\infty} G(t, s)|g(s, u_n(s)) - g(s, u(s))| \, ds
\]

\[
= \frac{1}{1+t} \int_0^{+\infty} G(t, s) \left| g \left( s, p_1(s) \left( \frac{u_n(s)}{p_1(s)} \right) \right) \right| - g \left( s, p_1(s) \left( \frac{u(s)}{p_1(s)} \right) \right) \right| \, ds
\]

\[
\leq \frac{2}{p_1(t)} \int_0^{+\infty} G(t, s)\psi_R(s) \, ds
\]

\[
\leq \|\psi_R\|_2 \sup_{t \geq 0} \left( \frac{2}{p_1(t)} \int_0^{+\infty} G(t, s) \, ds \right) = \frac{2\|\psi_R\|_2}{k^2}.
\]

Let \((t_n)\) be such that \( \Phi_n(t_n) = \sup_{t \geq 0} \Phi_n(t) \) and let \((t_{n_i})\) be such that \( \lim \Phi_n(t_{n_i}) = \lim sup \Phi_n(t_n) \). Therefore, we have to prove that \( \lim \Phi_n(t_{n_i}) = 0 \). We distinguish then two cases:

i) \((t_{n_i})\) is bounded by \( c > 0 \): In this case we have

\[
\Phi_n(t_{n_i}) = \left( \frac{1}{p_1(t_{n_i})} \int_0^{+\infty} G(t_{n_i}, s)|g(s, u_{n_i}(s)) - g(s, u(s))| \, ds \right)
\]

\[
\leq \int_0^{+\infty} G(c, s)|g(s, u_{n_i}(s)) - g(s, u(s))| \, ds,
\]

\[
\lim_{n \rightarrow +\infty} G(c, s)|g(s, u_{n_i}(s)) - g(s, u(s))| = 0,
\]

\[
|g(s, u_{n_i}(s)) - g(s, u(s))| = \left| g \left( t, p_1(s) \left( \frac{u_{n_i}(s)}{p_1(s)} \right) \right) \right| - g \left( t, p_1(s) \left( \frac{u(s)}{p_1(s)} \right) \right) \right| \leq 2\psi_R(s),
\]

for all \( s > 0 \) and by (28) \( \int_0^{+\infty} G(c, s)\psi_R(s) \, ds < \infty \). Hence the dominated convergence theorem leads to \( \lim \Phi_n(t_{n_i}) = \lim sup \Phi_n(t_n) = 0 \).

ii) \( \lim t_{n_i} = +\infty \) (up to a subsequence): In this case we have from Lemma 2,

\[
\Phi_n(t_{n_i}) = \left( \frac{1}{p_1(t_{n_i})} \int_0^{+\infty} G(t_{n_i}, s)|g(s, u_{n_i}(s)) - g(s, u(s))| \, ds \right)
\]

\[
\leq \frac{2}{p_1(t_{n_i})} \int_0^{+\infty} G(t_{n_i}, s)\psi_R(s) \, ds \rightarrow 0 \quad \text{as } l \rightarrow \infty.
\]
Thus, we have proved that \( \lim T^1_{g_0}u_{n_i} = T^1_{g_0}u \) in \( E_1 \) and \( T^1_{g_0} \) is continuous.

Now we prove by means of Lemma 4 that \( T^1_{g_0} \) maps bounded sets of \( E_1 \) into relatively compact sets of \( E_1 \). To this aim, let \( \Omega \) be a subset of \( E_1 \) bounded by \( R > 0 \) and let \( \psi_R \in \Gamma_1 \) be such that

\[
|g(s, p_1(s))u| \leq \psi_R(s) \text{ for all } s \geq 0 \text{ and all } u \in [-R, R].
\]

For any \( u \in \Omega \) we have by Property (29),

\[
\|T^1_{g_0}u\|_1 = \sup_{t \geq 0} \left| \frac{T^1_{g_0}u(t)}{p_1(t)} \right| = \sup_{t \geq 0} \left( \frac{1}{p_1(t)} \int_0^{+\infty} G(t, s) \left| g \left( s, p_1(s) \left( \frac{u(s)}{p_1(s)} \right) \right) \right| ds \right)
\]

\[
\leq \sup_{t \geq 0} \left( \frac{1}{p_1(t)} \int_0^{+\infty} G(t, s)\psi_R(s)ds \right)
\]

\[
\leq \sup_{t \geq 0} \left( \frac{1}{p_1(t)} \int_0^{+\infty} G(t, s)ds \right) \|\psi_R\|_1 = \frac{1}{k^2} \|\psi_R\|_1.
\]

Hence \( T^1_{g_0}(\Omega) \) is bounded in \( E_1 \).

Let \( t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}^+ \) with \( t_1 \leq t_2 \). For all \( u \in \Omega \) we have

\[
\left| \frac{T^1_{g_0}u(t_2)}{p_1(t_2)} - \frac{T^1_{g_0}u(t_1)}{p_1(t_1)} \right| \leq \int_0^{t_1} \left| \frac{G(t_2, s)}{p_1(t_2)} - \frac{G(t_1, s)}{p_1(t_1)} \right| \psi_R(s)ds + \int_{t_1}^{t_2} \left| \frac{G(t_2, s)}{p_1(t_2)} - \frac{G(t_1, s)}{p_1(t_1)} \right| \psi_R(s)ds
\]

\[
+ \int_0^{+\infty} \left| \frac{G(t_2, s)}{p_1(t_2)} - \frac{G(t_1, s)}{p_1(t_1)} \right| \psi_R(s)ds,
\]

\[
\int_0^{t_1} \left| \frac{G(t_2, s)}{p_1(t_2)} - \frac{G(t_1, s)}{p_1(t_1)} \right| \psi_R(s)ds \leq \frac{1}{k^2} \left( e^{-kt_1} - e^{-kt_1} \right) \int_0^{\zeta} \sinh(ks)\psi_R(s)ds
\]

\[
+ \frac{1}{k^2} \left( \frac{1}{p_1(t_1)} - \frac{1}{p_1(t_2)} \right) \int_0^{\zeta} (1 - e^{-ks})\psi_R(s)ds
\]

\[
\leq \frac{C_1(k)}{k^2} \left( \int_0^{\zeta} \psi_R(s)ds \right) (t_2 - t_1),
\]

\[
\int_{t_1}^{t_2} \left| \frac{G(t_2, s)}{p_1(t_2)} - \frac{G(t_1, s)}{p_1(t_1)} \right| \psi_R(s)ds \leq \frac{1}{k^2} \int_{t_1}^{t_2} \left( e^{-kt_2} - e^{-kt_1} \right) \frac{\sinh(ks)}{p_1(t_2)} + \frac{1 - e^{-ks}}{p_1(t_2)} + \frac{\cosh(kt_1) - 1}{p_1(t_1)} \psi_R(s)ds
\]

\[
\leq \frac{C_2(k)}{k^2} (t_2 - t_1)
\]

and

\[
\int_{t_2}^{+\infty} \left| \frac{G(t_2, s)}{p_1(t_2)} - \frac{G(t_1, s)}{p_1(t_1)} \right| \psi_R(s)ds \leq \frac{1}{k^2} \left( \frac{\cosh(kt_2) - 1}{p_1(t_2)} - \frac{\cosh(kt_1) - 1}{p_1(t_1)} \right) \int_\eta^{+\infty} e^{-ks}\psi_R(s)e^{-ks}ds
\]

\[
\leq \frac{C_3(k)}{k^2} (t_2 - t_1),
\]

where

\[
C_1(k) = (k + 1) \sinh(k\zeta) + 1,
\]

\[
C_2(k) = \left( \frac{\sinh(k\zeta)e^{-kn}}{1 + \eta} + 1 + \frac{\cosh(k\zeta) - 1}{1 + \zeta} \right) \sup_{s \in [\eta, \zeta]} \psi_R(s),
\]

\[
C_3(k) = \sup_{t \in [\eta, \zeta]} \left( \frac{\cosh(kt) - 1}{1 + t} \right)'.
\]
We obtain from the above computations that
\[
\left| \frac{T^1_g u(t_2)}{p_1(t_2)} - \frac{T^1_g u(t_1)}{p_1(t_1)} \right| \leq C_1(k) + C_2(k) + C_3(k) \frac{t_2 - t_1}{k^2}.
\]

Hence \( T^1_g (\Omega) \) is equicontinuous on compact intervals of \( \mathbb{R}^+ \).

We have for all \( u \in \Omega \) and \( t \geq 0 \)
\[
\frac{|T^1_g u(t)|}{1 + t} \leq \int_0^{+\infty} G(t, s) \frac{g(s, u(s))}{1 + t} ds \leq \frac{1}{1 + t} \int_0^{+\infty} G(t, s)\psi_R(s)ds := \bar{H}(t).
\]

Since Lemma 2 guarantees that \( \lim_{t \to +\infty} \bar{H}(t) = 0 \), we conclude that \( T^1_g (\Omega) \) is equiconvergent at \( +\infty \). This ends the proof.

**Lemma 6** Let \( g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) be a \( \Gamma_2 \)-Caratheodory function. Then the operator \( T^2_g \) is completely continuous.

**Proof.** First, let us prove that \( T^2_g \) is continuous. To this aim let \( (u_n) \) be a sequence in \( E_2 \) with \( \lim u_n = u \) in \( E_2 \), and let \( R > 0 \) and \( \psi_R \) be such that \( \|u_n\|_2 \leq R \) for all \( n \geq 1 \) and \( |g(t, p_2(t)u)| \leq \psi_R(t) \) for all \( t \geq 0 \) and \( u \in [-R, R] \).

Hence we have
\[
\|T^2_g u_n - T^2_g u\|_2 = \sup_{t \geq 0} |T^2_g u_n(t) - T^2_g u(t)| \leq \int_0^{+\infty} G(\infty, s)|g(s, u_n(s)) - g(s, u(s))| ds
\]

with
\[
\lim_{n \to +\infty} |g(s, u_n(s)) - g(s, u(s))| = 0
\]

and
\[
|g(s, u_n(s)) - g(s, u(s))| = |g(s, p_2(s)u_n(s)) - g(s, p_2(s)u(s))| \leq 2\psi_R(s).
\]

for all \( s > 0 \). Since \( \psi_R \in L^1(\mathbb{R}^+) \), we conclude by means of the dominated convergence theorem that \( \lim T^2_g u_n = T^2_g u \) in \( E_2 \), proving the continuity of \( T^2_g \).

Now we prove by means of Lemma 4 that \( T^2_g \) maps bounded sets of \( E_2 \) into relatively compact sets of \( E_2 \). To this aim, let \( \Omega \) be a subset of \( E_2 \) bounded by a constant \( R > 0 \) and let \( \psi_R \in \Gamma_2 \) be such that
\[
|g(s, p_2(s)u)| \leq \psi_R(s) \text{ for all } s \geq 0 \text{ and all } u \in [-R, R].
\]

Hence for all \( u \in \Omega \), we have by Property (25) and (27)
\[
\|T^2_g u\|_2 \leq \sup_{t \geq 0} \int_0^{+\infty} G(t, s)|g(s, u(s))| ds = \sup_{t \geq 0} \int_0^{+\infty} G(t, s)|g(s, p_2(s)u(s))| ds
\]
\[
\leq \int_0^{+\infty} G(\infty, s)\psi_R(s)ds < \infty.
\]

This estimate proves that \( T^2_g (\Omega) \) is bounded in \( E_2 \).

Let \( t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}^+ \) and \( u \in \Omega \). By Property (30) of the function \( G \), we obtain
\[
|T^2_g u(t_2) - T^2_g u(t_1)| \leq \int_0^{+\infty} |G(t_2, s) - G(t_1, s)| ds \|\psi_R\|_1 \leq \frac{2\|\psi_R\|_1}{k^2} |t_2 - t_1|.
\]

Proving that \( T^2_g (\Omega) \) is equicontinuous on compact intervals of \( \mathbb{R}^+ \).

We have for all \( u \in \Omega \) and \( t \geq 0 \)
\[
|T^2_g u(\infty) - T^2_g u(t)| \leq \int_0^{+\infty} (G(\infty, s) - G(t, s)) \psi_R(s)ds := H(t).
\]
Taking in account Property (27) and the fact that
\[
(G(\infty, s) - G(t, s)) \psi_R(s) \leq \frac{1}{k^2} \psi_R(s) \text{ for all } s > 0,
\]
where \( \psi_R \in L^1(\mathbb{R}^+) \), we obtain by the dominated convergence theorem that \( \lim_{t \to +\infty} H(t) = 0 \). Thus \( T_2^3(\Omega) \) is equiconvergent at \( +\infty \) and the proof is complete. \( \blacksquare \)

**Lemma 7** Assume the function \( g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) is \( \Gamma_3 \)-Caratheodory with \( i = 1 \) or \( 2 \). Then the operator \( T_2^3 \) is completely continuous.

**Proof.** Observe that since \( g \) is \( \Gamma_3 \)-Caratheodory, for all \( u \in E_3 \) we have \( T_2^3 u \in E_1 \). Therefore considering the operator \( T_2^{1,3} : E_3 \to E_1 \) with \( T_2^{1,3} u(t) = T_2^3 u(t) \) and arguing as in the proofs of Lemmas 5, we obtain that \( T_2^{1,3} \) is completely continuous. Since \( T_2^3 = I_1 \circ T_2^{1,3} \), where \( I_1 \) is the continuous embedding of \( E_1 \) in \( E_3 \), we have that \( T_2^3 \) is completely continuous. \( \blacksquare \)

We obtain from Lemmas 5, 6 and 7 the following fixed point formulation for the bvp (6).

**Corollary 4** Suppose that the function \( f \) is \( \Gamma_i \)-Caratheodory for some \( i \in \{1, 2, 3\} \). Then \( u_i \in E_i \) is a positive solution to the bvp (6) if and only if \( u_i \) is a fixed point of \( T_j^i \) where \( T_j^i : K_i \to K_i \) is completely continuous.

### 5 Proofs of Main Results

#### 5.1 Auxiliary Results

Let for \( q \in \Delta_i \) with \( i = 1, 2, 3 \), \( L_q^i : E_i \to E_i \) be the linear operator defined by
\[
L_q^i u(t) = \int_0^{+\infty} G(t, s) q(s) u(s) ds \text{ for } u \in E_i.
\]

We have from Lemmas 5, 6 and 7 that for \( i = 1, 2, 3 \), the linear operator \( L_q^i \) is compact. The main goal of this subsection is to prove that for \( i = 1, 2, 3 \), the operator \( L_q^i \) has the SJP at its spectral radius \( r(L_q^i) \) and in particular, \( L_q^3 \) is lower bounded on \( K_3 \). These results are requirement of Proposition 3, Theorem 5 and Theorem 6, and so are needed for the proofs of the main results of this article. We start by introducing some notations.

Let for \( T > 0 \), \( G_T : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) be the function defined by
\[
G_T(t, s) = \begin{cases} 
G(t, s), & \text{if } t \leq T, \\
G(T, s), & \text{if } t > T.
\end{cases}
\]

and for \( i = 1, 2, \)
\[
E_T = \{ u \in C(\mathbb{R}^+) : u(0) = 0 \text{ and } u(t) = u(T) \text{ for } t \geq T \},
\]
\[
X_T = \{ u \in E_T \cap C^2[0, T] : u'(0) = 0 \},
\]
\[
Y_T = X_T \cap C^3[0, T].
\]

Equipped respectively with the norms
\[
\| u \|_T = \sup_{t \in [0, T]} |u(t)| \text{ for all } u \in E_T,
\]
\[
\| u \|_X = \max(\| u \|_T, \| u' \|_T, \| u'' \|_T) \text{ for all } u \in X_T
\]
and
\[
\| u \|_Y = \max(\| u \|_X, \| u'' \|_T) \text{ for all } u \in Y_T,
\]
$E_T$, $X_T$ and $Y_T$ become Banach spaces.

In what follows $E^+_T$ and $X^+_T$ denote respectively the cones of nonnegative functions in the Banach spaces $E_T$ and $X_T$. For $q \in \Delta$ and $T > 0$, let $L^i_{q,T} : E_i \to E_i$, $L_{q,T} : E_T \to E_T$, $A_{q,T} : X_T \to X_T$, $\tilde{L}_{q,T} : E_T \to Y_T$, and $\tilde{A}_{q,T} : X_T \to Y_T$ be the linear bounded operators defined by

$$L^i_{q,T}u(t) = \int_0^{+\infty} G_T(t,s)q(s)u(s)ds \text{ for } u \in E_i,$$

$$\tilde{L}_{q,T}u = L_{q,T}u = L^i_{q,T}u \text{ for } u \in E_T$$

and

$$A_{q,T}u(t) = \tilde{A}_{q,T}u = L_{q,T}u \text{ for } u \in X_T.$$

Let $I, J$ be respectively the compact embedding of $Y_T$ into $E_T$ and $Y_T$ into $X_T$. Since $L_{q,T} = I \circ \tilde{L}_{q,T}$ and $A_{q,T} = J \circ \tilde{A}_{q,T}$, we have that $L_{q,T}$ and $A_{q,T}$ are compact operators. Moreover, arguing as in the proofs of Lemmas 5 and 6, we obtain that for $i = 1, 2$, $L^i_{q,T}$ is a compact operator.

**Lemma 8** The set $O_T$ defined by

$$O_T = \{u \in X_T : u' > 0 \text{ in } (0,T] \text{ and } u''(0) > 0\},$$

is open in the Banach space $X_T$.

**Proof.** We have $O^c_T = F_1 \cup F_2$ where

$$F_1 = \{u \in X_T : u'(t_0) \leq 0 \text{ for some } t_0 \in (0,T]\},$$

$$F_2 = \{u \in X_T : u''(0) \leq 0\}.$$

Since $F_2$ is a closed set in $X_T$, we have to show that $F^c_1 \subset F_1 \cup F_2$. To this aim, let $(u_n) \subset F_1$ with $\lim u_n = u$ and let $(x_n) \subset (0,T]$ be such $u'(x_n) \leq 0$ and $\lim x_n = \pi$. We distinguish the following two cases:

**Case 1.** $\pi \in (0,T]$: In this case we have

$$u'(\pi) = \lim u_n'(x_n) \leq 0,$$

proving that $u \in F_1$.

**Case 2.** $\pi = 0$: In this case we have

$$u''(0) = \lim_{n \to \infty} \frac{u_n'(x_n)}{x_n} \leq 0,$$

proving that $u \in F_2$. ■

**Lemma 9** For $i = 1$ or $2$, $q$ in $\Delta_i$ and $T > 0$, the operator $L^i_{q,T}$ has the SIPP at its spectral radius $r(L^i_{q,T})$.

**Proof.** First, we show that the linear mapping $A_{q,T}$ is strongly positive. Let $u \in X^+_T \setminus \{0\}$ and $v = A_{q,T}u$, we have from Property (25) of the function $G$ that

$$v'(t) = \int_0^T \frac{\partial G_T}{\partial t}(t,s)q(s)u(s)ds > 0 \text{ for all } t \in (0,T).$$

Moreover, we have

$$v''(0) = \int_0^T \frac{\partial^2 G_T}{\partial t^2}(t,s)q(s)u(s)ds > 0.$$  \hspace{1cm} (35)

Clearly, (35) and (36) show that $v = A_{q,T}u \in O_T \subset \text{int } (X^+_T)$, proving that

$$A_{q,T}(X^+_T \setminus \{0\}) \subset O_T \subset \text{int } (X^+_T) \text{ and } A_{q,T} \text{ is strongly positive.}$$
Therefore, we conclude from Proposition 2 that the operator $A_{q,T}$ has the SIJP at $r(A_{q,T})$.

Now, we are able to prove that the operator $L_{q,T}$ has the SIJP at $r(L_{q,T})$. Let $\mu_0 > 0$ and $u \in E_T^+ \setminus \{0\}$ such that $L_{q,T}u \geq \mu_0 u$, then $U = L_{q,T}u \in X_T^+ \setminus \{0\}$ and satisfies $L_{q,T}U = A_{q,T}U \geq \mu_0 U$. Hence, we have that $\mu_0 \in \Lambda_{A_{q,T}}$ and $\mu_0 \leq \sup \Lambda_{A_{q,T}} = r(A_{q,T})$.

Similarly if $\eta_0 \geq 0$ and $v \in E_T^+ \setminus \{0\}$ are such that $L_{q,T}v \leq \eta_0 v$, then $V = L_{q,T}v \in X_T^+ \setminus \{0\}$ and satisfies $L_{q,T}V = A_{q,T}V \leq \eta_0 V$. Therefore, we have that

$$\eta_0 \in \Gamma_{A_{q,T}} \text{ and } \eta_0 \geq \inf \Gamma_{A_{q,T}} = r(A_{q,T}).$$

Therefore, we have proved that

$$\sup \Lambda_{L_{q,T}} \leq r(A_{q,T}) = \inf \Gamma_{A_{q,T}} = \sup \Lambda_{A_{q,T}} \leq \inf \Gamma_{L_{q,T}}$$

and this combined with (17) leads to $\inf \Gamma_{L_{q,T}} = \sup \Lambda_{L_{q,T}} = r(A_{q,T})$ and $L_{q,T}$ has the SIJP at $r(A_{q,T})$.

Since the cone $E_T^+$ is total in the Banach space $E_T$, we have that $r(L_{q,T})$ is a positive eigenvalue. Hence taking in consideration Remark 1, we obtain that $r(L_{q,T}) = r(A_{q,T})$ and $L_{q,T}$ has the SIJP at $r(L_{q,T})$.

Noticing that for all $u \in K_i \setminus \{0\}$, $X_T^+ \setminus \{0\}$ and $L_{q,T}U = L_{q,T}U$, then arguing as above we obtain that $L_{q,T}^1$ has the SIJP at $r(L_{q,T}^1)$. Ending the proof. \]

**Theorem 7** For $i = 1$ or $2$ and $q$ in $\Delta$, the operator $L_{q}^i$ has the SIJP at its spectral radius $r(L_{q}^i)$.

**Proof.** In order to make use of Theorem 4 we prove that for a function $q$ in $\Delta$, $T \to L_{q,T}^i$ is increasing and $\lim_{T \to +\infty} L_{q,T}^i = L_{q}^i$. Let $q$ in $\Delta_i$ and $T_1, T_2$ be such that $0 < T_1 < T_2 < +\infty$. For $u \in K_i$ we have

$$L_{q,T}^i u(t) - L_{q,T}^1 u(t) = \begin{cases} \int_{0}^{+\infty} (G(t, s) - G(t, T)) q(s) u(s) ds = 0, & \text{if } t \leq T_1, \\ \int_{0}^{+\infty} (G(t, T_1) - G(T, T_1)) q(s) u(s) ds \geq 0, & \text{if } T_1 < t \leq T_2, \\ \int_{0}^{+\infty} (G(T_2, s) - G(T_1, s)) q(s) u(s) ds \geq 0, & \text{if } T_2 < t, \end{cases}$$

proving that $L_{q,T}^i u - L_{q,T}^1 u \in K_i$ and $L_{q,T}^i \leq L_{q,T}^i$.

If $i = 1$, for $u \in E_1$ with $\|u\|_1 = 1$, we have

$$\left| \frac{L_{q}^1 u(t) - L_{q,T}^1 u(t)}{p_1(t)} \right| = \begin{cases} 0, & \text{if } t \leq T, \\ \frac{1}{1 + t} \int_{0}^{+\infty} (G(t, s) - G(T, T)) q(s) ds, & \text{if } t \geq T. \end{cases}$$

Therefore,

$$\sup_{t \geq 0} \left| \frac{L_{q}^1 u(t) - L_{q,T}^1 u(t)}{1 + t} \right| = \sup_{t \geq T} \left( \frac{1}{1 + t} \int_{0}^{+\infty} (G(t, s) - G(T, T)) q(s) ds \right) \leq \sup_{t \geq T} \left( \frac{1}{1 + t} \int_{0}^{+\infty} G(t, s) q(s) ds \right).$$

Since

$$\lim_{t \to +\infty} \left( \frac{1}{1 + t} \int_{0}^{+\infty} G(t, s) q(s) ds \right) = 0,$$

we have

$$\lim_{T \to +\infty} \left( \sup_{\|u\|_1 = 1} \left\| L_{q}^1 u - L_{q,T}^1 u \right\|_1 \right) = \lim_{T \to +\infty} \left( \sup_{\|u\|_1 = 1} \left( \sup_{t \geq 0} \left| \frac{L_{q}^1 u(t) - L_{q,T}^1 u(t)}{1 + t} \right| \right) \right) \leq \lim_{T \to +\infty} \left( \sup_{t \geq T} \left( \frac{1}{1 + t} \int_{0}^{+\infty} G(t, s) q(s) ds \right) \right) = 0.$$
Hence we obtain by Theorem 4 that the operator $L^i_q$ has the SIJP at its spectral radius $r(L^i_q)$.

If $i = 2$, for $u \in E_2$ with $\|u\|_2 = 1$ we have

$$|L^2_q u(t) - L^2_{q,T} u(t)| \leq \int_0^{+\infty} (G(t, s) - G_T(t, s)) q(s) ds$$

$$= \begin{cases} 0, & \text{if } t \leq T, \\ \int_0^{+\infty} (G(t, s) - G(T, s)) q(s) ds, & \text{if } t \geq T. \end{cases}$$

Hence we have

$$\|L^2_q - L^2_{q,T}\| = \sup_{\|u\|_2 = 1} \|L^2_q u - L^2_{q,T} u\| \leq \int_0^{+\infty} (G(t, s) - G(T, s)) q(s) ds,$$

then by Lebesgue dominated convergence theorem we conclude that $L^2_{q,T} \to L^2_q$ as $T \to +\infty$. By Theorem 4, we obtain that the operator $L^2_q$ has the SIJP at its spectral radius $r(L^2_q)$. ■

**Theorem 8** For $i = 1$ or 2 and $q$ in $\Delta_3$ the operator $L^3_q$ has the SIJP at its spectral radius $r(L^3_q)$ and $L^3_q$ is bounded on the cone $K_3$ from below.

**Proof.** Notice first that for all $u \in K_3$, $L^3_q u \in K_1$. Indeed, we have for $u \in K_3$ and for all $t > 0$

$$\frac{L^3_q u(t)}{1 + t} \leq \|u\|_3 \|L^3_q u - L^3_{q,T} u\| \leq \int_0^{+\infty} G(t, s) (e^{ks} q(s)) ds \to 0 \text{ as } t \to +\infty,$$

since $\lim_{s \to +\infty} e^{ks} q(s) = 0$, and

$$(L^3_q u)'(t) = \int_0^{+\infty} \frac{\partial G}{\partial t}(t, s) q(s) u(s) ds > 0.$$ 

Let now, $\lambda_0 > 0$ and $u \in K_3 \setminus \{0\}$ be such that $L^3_q u \leq \lambda_0 u$. Then $U = L^3_q u$ satisfies $L^3_q U = L^3_q U \leq \lambda_0 U$ and we have $\lambda_0 \geq \inf \Gamma_{L^3_q} = r(L^3_q)$. Similarly if $\theta_0 > 0$ and $u \in K_3 \setminus \{0\}$ are such that $L^3_q u \geq \theta_0 u$ then $U = L^3_q u \in K_1 \setminus \{0\}$ and satisfies $L^3_q U = L^3_q U \geq \theta_0 U$ and we have $\theta_0 \leq \sup \Lambda_{L^3_q} = r(L^3_q)$.

The above leads to $r(L^3_q) = \inf \Gamma_{L^3_q} = \sup \Lambda_{L^3_q}$ and the operator $L^3_q$ has the SIJP at $r(L^3_q)$. Since the cone $K_3$ is total in the Banach space $E_3$ and Remark 1 claims that $r(L^3_q)$ is the unique positive eigenvalue of the positive operator $L^3_q$, we have that $r(L^3_q) = r(L^3_q)$ and $L^3_q$ has the SIJP at $r(L^3_q)$.

It remains to show that $L^3_q$ is lower bounded on $K_3$. Let $u \in K_3$, with $\|u\|_3 = 1$, we have then for all $t \geq 0$,

$$L^3_q u(t) = \int_0^{+\infty} G(t, s) q(s) u(s) ds \geq \int_0^{+\infty} G(t, s) q(s) \gamma(s) ds,$$

leading to

$$\inf \left\{ \|L^3_q u\|_3 : u \in K_3 \cap \partial B(0_{E_3}, 1) \right\} \geq \sup_{t \geq 0} e^{-kt} \int_0^{+\infty} G(t, s) q(s) \gamma(s) ds > 0$$

and the operator $L^3_q$ is lower bounded on the cone $K_3$ from below. This ends the proof. ■

### 5.2 Proof of Proposition 1

Let $q \in \Delta$, we have from Lemma 2 that $\mu$ is a positive eigenvalue of the linear eigenvalue problem (7) if and only if $\mu^{-1}$ is a positive eigenvalue of the compact operator $L^i_q$ for $i = 1$ or 2. Since Theorem 7 claims that $L^i_q$ has the SIJP at $r(L^i_q)$, we have from Remark 1 that $r(L^i_q)$ is the unique positive eigenvalue of $L^i_q$. Therefore, we have that $\mu(q) = 1/r(L^i_q)$ is the unique positive eigenvalue of the linear eigenvalue problem (7).
Now, let \( \phi \) be the eigenfunction associated with \( \mu(q) \). Clearly if \( q \in \Delta_2 \) then \( \phi \) is bounded and if not then \( \phi \) satisfies
\[
\phi(t) = \int_0^t G(t, s) q(s) \phi(s) ds \geq \frac{1}{k^2} \int_1^t \left( -e^{-k s} \sinh(ks) + (1 - e^{-ks}) \right) q(s) \phi(s) ds
\]
\[
\geq \frac{(1 - e^{-k})^2}{2k^2} \int_1^t q(s) \phi(s) ds
\]
\[
\geq \frac{(1 - e^{-k})^2}{2k^2} \phi(1) \int_1^t q(s) ds.
\]
(37)
Thus, suppose to the contrary that \( \phi \) is bounded, then passing to the limits in (37), we obtain the contradiction
\[
+\infty > \lim_{t \to +\infty} \phi(t) = \lim_{t \to +\infty} \frac{(1 - e^{-k})^2}{2k^2} \phi(1) \int_1^t q(s) ds = +\infty.
\]
Ending the proof.

5.3 Proof of Theorem 1
Assume that Hypothesis (8) holds true (the case where (9) holds is checked similarly). Let \( \epsilon > 0 \) be so small such that for \( i = 1, 2, \)
\[
\inf \left\{ \frac{f(t, p_i(t)u)}{p_i(t)q(t)u} : t, u > 0 \right\} \geq (\mu(q) + \epsilon).
\]
Hence for all \( u \in K_i \), we have
\[
T_i^I u(t) = \int_0^t G(t, s) f(s, u(s)) ds
\]
\[
= \int_0^t G(t, s) f(s, p_i(s) \frac{u(s)}{p_i(s)}) ds
\]
\[
\geq (\mu(q) + \epsilon) \int_0^t G(t, s) q(s) u(s) ds
\]
\[
= (\mu(q) + \epsilon) L^I u(t) \equiv L^I_q u(t)
\]
and
\[
r(\tilde{L}^I_q) = \frac{\mu(q) + \epsilon}{\mu(q)} > 1.
\]
Since Theorems 7 and 8 state that the operator \( \tilde{L}^I_q \) has the SLJP at \( r(\tilde{L}^I_q) \), Hypothesis (18) holds and Proposition 3 guarantees that the operator \( T_i^I \) has no fixed point in \( K_i \). Thus, we conclude by Corollary 4 that the bvp (6) has no positive solution.

5.4 Proof of Theorem 2
Step 1. Existence in the case where (10) is satisfied
Let \( \epsilon \in (0, \mu(q) - f^+_{i, +\infty} (q_{\infty})) \) there is \( R \) such that
\[
f(t, p_i(t)u) \leq (\mu(q) - \epsilon) p_i(t) q_{\infty}(t) u \text{ for all } t \geq 0 \text{ and } u \geq R.
\]
Since the function \( f \) is \( \Gamma_i \)-Caratheodory, there is \( \psi_R \in \Gamma_i \) such that
\[
f(t, p_i(t)u) \leq (\mu(q) - \epsilon) p_i(t) q_{\infty}(t) u + \psi_R(t) \text{ for all } t, u \geq 0,
\]
and this leads to
\[ f(t, u) \leq (\mu(q_\infty) - \varepsilon) q_\infty(t) u + \psi_R(t) \text{ for all } t, u \geq 0. \] (38)

Let \( \varepsilon \in (0, f^-_{i,0}(q_0) - \mu(q_\infty)) \) there is \( r > 0 \) such that for all \( t \geq 0 \) and \( u \in [0, r] \)
\[ (f^-_{i,0}(q_0) + \varepsilon) p_i(t) q_0(t) u \geq f(t, p_i(t) u) \geq (\mu(q_\infty) + \varepsilon) p_i(t) q_0(t) u, \]
leading to
\[ (f^-_{i,0}(q_0) + \varepsilon) q_0(t) u \geq f(t, u) \geq (\mu(q_\infty) + \varepsilon) q_0(t) u \text{ for all } t \geq 0 \text{ and } u \in [0, r]. \]
Therefore, for all \( t, u \geq 0 \) we have
\[ (f^-_{i,0}(q_0) + \varepsilon) q_0(t) u + \tilde{f}(t, u) \geq f(t, u) \geq (\mu(q_0) + \varepsilon) q_0(t) u - \tilde{f}(t, u), \] (39)
where
\[ \tilde{f}(t, u) = \sup (0, (\mu(q_\infty) + \varepsilon) q_0(t) u - f(t, u)), \]
\[ \hat{f}(t, u) = \sup (0, f(t, u) - (f^-_{i,0}(q_0) + \varepsilon) q_0(t) u). \]
Therefore, we obtain from (38) and (39) that
\[ T^i u \leq L^i u + F_\infty u \text{ for all } u \in K_i, \]
and
\[ L^i q_0 u - F_0 u \leq T^i u \leq L^i q_0 u + \hat{F}_0 u \text{ for all } u \in K_i \]
where
\[ F_0 u(t) = \int_0^{+\infty} G(t, s) f_0(u(s)) ds, \]
\[ \hat{F}_0 u(t) = \int_0^{+\infty} G(t, s) \hat{f}(t, u(s)) ds, \]
\[ F_\infty u(t) = \int_0^{+\infty} G(t, s) \psi_R(s) ds, \]
\[ r(L^i q_0) = \frac{(\mu(q_\infty) - \varepsilon)}{\mu(q_\infty)} < 1, \quad r(L^i q_0) = \frac{(\mu(q_0) + \varepsilon)}{\mu(q_0)}. \]

We conclude from Theorem 7, Theorem 5 and Corollary 4 that the bvp (6) admits a positive solution \( u \in K_i \).

**Step 2. Existence in the case where (11) is satisfied**

Let \( \varepsilon \in (0, \mu_i(q_0) - f^-_{i,0}(q_0)) \) there is \( r > 0 \) small such that
\[ f(t, p_i(t) u) \leq (\mu(q_\infty) - \varepsilon) p_i(t) q_\infty(t) u \text{ for all } t \geq 0 \text{ and } u \leq r, \]
leading to
\[ f(t, u) \leq (\mu(q_0) - \varepsilon) q_0(t) u \text{ for all } t \geq 0 \text{ and } u \leq r. \]
Therefore, for all \( t, u \geq 0 \) we have
\[ f(t, u) \leq (\mu(q_0) - \varepsilon) q_0(t) u + \tilde{f}(t, u), \] (40)
with
\[ \tilde{f}(t, u) = \sup (0, f(t, u) - (\mu(q_0) - \varepsilon) q_0(t) u). \]
Let \( \varepsilon \in (0, f^-_{i,\infty}(q_\infty) - \mu_i(q_\infty)) \) there is \( R > 0 \) such that for all \( t \geq 0 \) and \( u \geq R, \)
\[ (\mu(q_\infty) + \varepsilon) p_i(t) q_\infty(t) u \leq f(t, p_i(t) u) \leq (f^-_{i,\infty}(q_\infty) + \varepsilon) p_i(t) q_\infty(t) u, \]
Since the nonlinearity $f$ is a $\Gamma_i$-Caratheodory function, there is $\psi_R \in \Gamma_i$ such that

$$f(t, u) \leq (f^+_{i,\infty}(q_\infty) + \varepsilon) q_\infty(t) p_i(t) u + \psi_R(t) \quad \text{for all } t, u \geq 0.$$ 

Therefore, for all $t, u \geq 0$ we have

$$(\mu_i(q_\infty) + \varepsilon) q_\infty(t) u - \tilde{f}(t, u) \leq f(t, u) \leq (f^+_{i,\infty}(q_\infty) + \varepsilon) q_\infty(t) u + \psi_R(t),$$

where

$$\tilde{f}(t, u) = \sup (0, (\mu(q_\infty) + \varepsilon) q_\infty(t) u - f(t, u)).$$

Therefore, we obtain from (40) and (41) that

$$T^i J u \leq L^i_{q_0} u + F_0 u \text{ for all } u \in K_i$$

and

$$L^i_{q_\infty} u - F_\infty u \leq T^i J u \leq L^i_{q_\infty} u + \tilde{F}_\infty u \text{ for all } u \in K_i,$$

where

$$F_0 u(t) = \int_0^{+\infty} G(t, s) \tilde{f}(t, u(s)) ds,$$

$$\tilde{F}_\infty u(t) = \int_0^{+\infty} G(t, s) \psi_R(s) ds,$$

$$F_\infty u(t) = \int_0^{+\infty} G(t, s) f(t, u(s)) ds,$$

$$r\left(L^i_{q_0}\right) = \frac{(\mu(q_\infty) - \varepsilon)}{\mu(q_\infty)} < 1 < r\left(L^i_{q_\infty}\right) = \frac{(\mu(q_0) + \varepsilon)}{\mu(q_0)}.$$ 

We conclude from Theorem 7, Theorem 5 and Corollary 4 that the bvp (6) admits a positive solution $u \in K_i$.

**Step 3. Boundedness and unboundedness of the solution**

Evidently, if $i = 1$ the solution $u$ is bounded. If $i = 2$ and Hypothesis (12) is fulfilled, then the solution $u$ satisfies

$$u(t) = \int_0^{+\infty} G(t, s) f(s, u(s)) ds \geq \frac{(1 - e^{-k})^2}{2k^2} \int_1^t f(s, u(s)) ds = \frac{(1 - e^{-k})^2}{2k^2} \int_1^t f(s, p_1(s) \left(\frac{u(s)}{p_1(s)}\right)) ds.$$ 

Thus, suppose to the contrary that the solution $u$ is bounded, then passing to the limits in (42), we obtain the contradiction

$$+\infty \geq \lim_{t \to +\infty} u(t) = \lim_{t \to +\infty} \frac{(1 - e^{-k})^2}{2k^2} \int_1^t f(s, p_1(s) \left(\frac{u(s)}{p_1(s)}\right)) ds = +\infty.$$ 

Ending the proof.

### 5.5 Proof of Theorem 3

**Step 1. Existence in the case where (13) is satisfied**

Let $\varepsilon \in (0, \mu(q_\infty) - f^+_{i,\infty}(q_\infty))$, there is $R$ such that

$$f(t, p_3(t) u) \leq (\mu_1(q_\infty) - \varepsilon) p_3(t) q_\infty(t) u \text{ for all } t \geq 0 \text{ and } u \geq R.$$ 

Since the nonlinearity $f$ is a $\Gamma_i$-Caratheodory function, there is $\psi_R \in \Gamma_1$ such that

$$f(t, p_3(t) u) \leq (\mu(q_\infty) - \varepsilon) p_3(t) q_\infty(t) u + \psi_R(t) \text{ for all } t, u \geq 0.$$ 

and this leads to

\[ f(t, u) \leq (\mu (q_\infty) - \epsilon) q_\infty(t)u + \psi_R(t) \text{ for all } t, u \geq 0. \tag{43} \]

Also, we have from \( f_{-0}^- (q_0) > \mu(q_0) \) that for \( \epsilon \in (0, f_{-0}^- (q_0) - \mu(q_\infty)) \) there is \( r > 0 \) such that

\[ f(t, p_3(t)u) \geq (\mu(q_\infty) + \epsilon) p_3(t)q_0(t)u \text{ for all } t \geq 0 \text{ and } u \in [0, r], \]

leading to

\[ f(t, u) \geq (\mu(q_\infty) + \epsilon) q_0(t)u \text{ for all } t \geq 0 \text{ and } u \in [0, r]. \]

Therefore we have

\[ f(t, u) \geq (\mu(q_0) + \epsilon) q_0(t)u - \tilde{f}(t, u) \text{ for all } t, u \geq 0, \tag{44} \]

where

\[ \tilde{f}(t, u) = \sup (0, (\mu(q_\infty) + \epsilon) q_0(t)u - f(t, u)). \]

Hence, we obtain from (43) and (44) that

\[ L_{q_0}^3 u - F_0 u \leq T_f^3 u \leq L_{q_\infty}^3 u + F_\infty u \text{ for all } u \in K_3, \]

where

\[ F_0 u(t) = \int_0^{+\infty} G(t, s) \tilde{f}(t, u(s))ds, \]

\[ F_\infty u(t) = \int_0^{+\infty} G(t, s) \psi_R(s)ds, \]

\[ r (L_{q_\infty}^3) = \frac{(\mu(q_\infty) - \epsilon)}{\mu(q_\infty)} < 1 < r (L_{q_0}^3) = \frac{(\mu(q_0) + \epsilon)}{\mu(q_0)}. \]

We conclude from Theorem 8, Theorem 6 and Corollary 4 that the bvp (6) admits a positive solution.

**Step 2. Existence in the case where (14) is satisfied**

Let \( \epsilon \in (0, \mu(q_0) - f_{3,0}^+(q_0)) \), there is \( r > 0 \) such that

\[ f(t, p_3(t)u) \leq (\mu(q_0) - \epsilon) p_3(t)q_0(t)u \text{ for all } t \geq 0 \text{ and } u \leq r. \]

Hence for all \( t, u \geq 0 \) we have

\[ f(t, u) \leq (\mu(q_0) - \epsilon) q_0(t)u + \tilde{f}(t, u), \tag{45} \]

where

\[ \tilde{f}(t, u) = \sup (0, (\mu(q_0) - \epsilon) q_0(t)u). \]

Let \( \epsilon \in (0, f_{-0}^- (q_0) - \mu(q_\infty)) \) there is \( R > 0 \) such that

\[ f(t, p_3(t)u) \geq (\mu(q_\infty) + \epsilon) p_3(t)q_\infty(t)u \text{ for all } t \geq 0 \text{ and } u \geq R, \]

leading to

\[ f(t, u) \geq (\mu(q_\infty) + \epsilon) q_\infty(t)u \text{ for all } t \geq 0 \text{ and } u \geq R. \]

Therefore, we have

\[ f(t, u) \geq (\mu(q_\infty) + \epsilon) q_\infty(t)u - \tilde{f}(t, u) \text{ for all } t, u \geq 0, \tag{46} \]

where

\[ \tilde{f}(t, u) = \sup (0, (\mu(q_\infty) + \epsilon) q_\infty(t)u - f(t, u)). \]

Hence, we obtain from (45) and (46) that

\[ L_{q_\infty}^3 u - F_\infty u \leq T_f^3 u \leq L_{q_0}^3 u + F_0 u \text{ for all } u \in K_3, \]
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where

\[ F_0 u(t) = \int_0^{+\infty} G(t, s) \tilde{f}(t, u(s)) ds, \]
\[ F_\infty u(t) = \int_0^{+\infty} G(t, s) \tilde{f}(t, u(s)) ds, \]

\[ r \left( L^3_{q_0} \right) = \frac{\mu(q_0) - \epsilon}{\mu(q_0)} < 1 < r \left( L^3_{q_\infty} \right) = \frac{\mu(q_\infty) + \epsilon}{\mu(q_\infty)}. \]

We conclude from Theorem 8, Theorem 6 and Corollary 4 that the bvp (6) admits a positive solution.

**Step 3. Boundedness and unboundedness of the solution**

Evidently, if \( f \) is a \( \Gamma_4 \)-Caratheodory function the solution \( u \) is bounded. If Hypothesis (15) is fulfilled, then the solution \( u \) satisfies

\[
 u(t) = \int_0^{+\infty} G(t, s) f(s, u(s)) ds \geq \frac{(1 - e^{-k})^2}{2k^2} \int_1^t f(s, u(s)) ds = \frac{(1 - e^{-k})^2}{2k^2} \int_1^t f(s, p_3(s) \left( \frac{u(s)}{p_3(s)} \right)) ds. \quad (47)
\]

Thus, by the contrary if the solution \( u \) is bounded then passing to the limits in (47) we obtain the contradiction

\[
 \lim_{t \to +\infty} u(t) = \lim_{t \to +\infty} \left( \frac{1 - e^{-k}}{2k^2} \int_1^t f(s, p_3(s) \left( \frac{u(s)}{p_3(s)} \right)) ds \right) = +\infty.
\]

Ending the proof.

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**References**


