Functional Equations Characterizing Certain Determinants And Permanents

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Abstract

General solution functions \( g, h, f : \mathbb{R}^2 \to \mathbb{R} \) of the following three functional equations

\[
\begin{align*}
    g(ux - vy, uy + v(x + y)) &= g(x, y)g(u, v), \\
    h(ux - vy, uy + v(x - y)) &= h(x, y)h(u, v), \\
    f(vx + uy, vy - ux) &= f(x, y)f(u, v)
\end{align*}
\]

are determined without any regularity assumptions on the unknown functions. These equations arise from the problem of characterizing certain determinants and permanents.

1 Introduction

It is easily seen that the \( 2 \times 2 \) determinant function defined by

\[
d(x, y) := \det \begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 - y^2
\]

satisfies the functional equations

\[
\begin{align*}
    f(ux + vy, uy + vx) &= f(x, y)f(u, v), \\
    f(ux - vy, uy - vx) &= f(x, y)f(u, v),
\end{align*}
\]

while the \( 2 \times 2 \) permanent function defined by

\[
p(x, y) := \text{per} \begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 + y^2
\]

satisfies the functional equation

\[
f(ux + vy, uy - vx) = f(x, y)f(u, v).
\]

The functional equation (1), as well as its pexiderized version, was solved by Chung and Sahoo \cite{2} in 2002, while the functional equation (3) was solved first in 2003 by Jung and Bae \cite{6}. The equation (3) was solved again in 2014 by Chung and Chang \cite{3} using a different technique. The functional equation (1) was extended to

\[
f(ux + vy, uy + vx, zw) = f(x, y, z)f(u, v, w)
\]

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and was solved by Chung and Sahoo \[2\] in 2002, while the functional equations (2) and (3) were, respectively, generalized to
\[
\begin{align*}
  f(ux - vy, uy - vx, zw) &= f(x, y, z) f(u, v, w), \\
  f(ux + vy, uy - vx, zw) &= f(x, y, z) f(u, v, w),
\end{align*}
\]
and were solved by Choi et al. \[1\] in 2016; see also \[4\], \[5\] for related problems. Closer analysis reveals that earlier techniques used for solving (1)–(5) are not sufficient to solve the pexiderized version of (3) which is
\[
f(ux + vy, uy - vx) = g(x, y) h(u, v).
\]
In \[8\], the authors solved, among other things, (6) using the unit circle and trigonometric transformation. Indeed, to solve (6), it suffices to solve its non–pexiderized version, viz.,
\[
f(vx + uy, vy - ux) = f(x, y) f(u, v).
\]
In another direction, from the identity
\[
\text{per} \left( \begin{array}{cc}
ux - vy & uy + v(x + y) \\
u(y + v(x + y)) & ux + u(x + y)
\end{array} \right) = \text{per} \left( \begin{array}{cc}
x & y \\
y & x + y
\end{array} \right) \text{per} \left( \begin{array}{cc}
u & v \\
u & u + v
\end{array} \right).
\]
If the $2 \times 2$ permanent function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by
\[
g(x, y) := \text{per} \left( \begin{array}{cc}
x & y \\
y & x + y
\end{array} \right) = x^2 + xy + y^2,
\]
then this function $g$ satisfies
\[
g(ux - vy, uy + v(x + y)) = g(x, y) g(u, v),
\]
while the functional equation
\[
h(ux - vy, uy + v(x - y)) = h(x, y) h(u, v)
\]
is satisfied by the $2 \times 2$ permanent function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by
\[
h(x, y) := \text{per} \left( \begin{array}{cc}
x & y \\
y & x - y
\end{array} \right) = x^2 - xy + y^2.
\]
It is thus a natural question to search for general solution functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ of the two functional equations (8) and (9). The objective of this work is to solve both (8) and (9) by employing a new technique which can also be applied to give a simple treatment of the functional equation (7), solved in \[8\].

\section{Main Results}

Our analysis involves the use of the exponential and $E : \mathbb{R} \to \mathbb{R}$ and multiplicative function $M : \mathbb{R} \to \mathbb{R}$ which are generically defined \[7\], Chapter 1], respectively, as general solutions of the following two functional equations
\[
E(x + y) = E(x) E(y), \quad M(xy) = M(x) M(y).
\]

\textbf{Theorem 1} \textit{General solution functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ of the functional equations (8) and (9) are given by}
\begin{itemize}
  \item for the functional equation (8): $g \equiv 0$, or $g \equiv 1$, or
  \[
  g(x, y) = M \left( \sqrt{x^2 + xy + y^2} \right) E(\theta_1);
  \]
\end{itemize}
\* for the functional equation (9): \( h \equiv 0 \), or \( h \equiv 1 \), or

\[
h(x, y) = M \left( \sqrt{x^2 - xy + y^2} \right) E(\theta_2),
\]

where \( M \) and \( E \) are the multiplicative and exponential functions defined above, and \( \theta_1 \) and \( \theta_2 \) are the angular coordinates of the polar coordinates of \((2x + y)/2, -\sqrt{3}y/2)\) and \((2x - y)/2, -\sqrt{3}y/2)\), respectively.

**Proof.** To solve (8), note first that if \( g \) is a constant function solution of (8), then either \( g \equiv 0 \) or \( g \equiv 1 \). Assume now that \( g \) is a non-constant solution of (8). Substituting \( x \) by \( x + y/\sqrt{3} \), \( y \) by \(-2y/\sqrt{3} \), \( u \) by \( u + v/\sqrt{3} \) and \( v \) by \(-2v/\sqrt{3} \) into (8), we have

\[
g \left( (u - vy + (uy + vx)/\sqrt{3}, -2(uy + vx)/\sqrt{3}) \right) = g \left( (x + y/\sqrt{3}, -2y/\sqrt{3}) \right) g \left( (u + v/\sqrt{3}, -2v/\sqrt{3}) \right).
\]

Define the function \( G : \mathbb{C} \to \mathbb{R} \) by

\[
G(x + iy) = g(x + y/\sqrt{3}, -2y/\sqrt{3}) \quad (x, y \in \mathbb{R}).
\]

Then for \( z = x + iy \), \( w = u + iv \in \mathbb{C} \), using (10) and (11), we get

\[
G(zw) = G(ux - vy + (uy + vx)/\sqrt{3}, -2(uy + vx)/\sqrt{3}) = g \left( (x + y/\sqrt{3}, -2y/\sqrt{3}) \right) g \left( (u + v/\sqrt{3}, -2v/\sqrt{3}) \right) = G(z)G(w).
\]

Writing complex numbers in polar form

\[
z = r_1(\cos \theta_1 + i \sin \theta_1), \quad w = r_2(\cos \theta_2 + i \sin \theta_2),
\]

where \( r_1, r_2 \in \mathbb{R}_{\geq 0} \), and the angular coordinates \( \theta_1, \theta_2 \in \mathbb{R} \) are considered mod 2\( \pi \), and substituting into (12), we get

\[
G(r_1r_2(\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2))) = G(r_1(\cos \theta_1 + i \sin \theta_1))G(r_2(\cos \theta_2 + i \sin \theta_2)).
\]

Define the function \( \bar{G} : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R} \) by

\[
\bar{G}(r, \theta) = G(r(\cos \theta + i \sin \theta)),
\]

where \( \theta \) is considered mod 2\( \pi \); note that the function \( \bar{G} \) is periodic mod 2\( \pi \) in the second variable \( \theta \). Using (13), we have

\[
\bar{G}(r_1r_2, \theta_1 + \theta_2) = \bar{G}(r_1, \theta_1) \bar{G}(r_2, \theta_2).
\]

Taking \( \theta_1 = \theta_2 = 0 \) in (15), we get

\[
\bar{G}(r_1r_2, 0) = \bar{G}(r_1, 0) \bar{G}(r_2, 0).
\]

Define the function \( M : \mathbb{R}_{\geq 0} \to \mathbb{R} \) by

\[
M(x) = \bar{G}(x, 0).
\]

Using (16) and (17), we deduce that \( M \) is multiplicative. In another direction, setting \( r_1 = r_2 = 1 \) in (15), we have

\[
\bar{G}(1, \theta_1 + \theta_2) = \bar{G}(1, \theta_1) \bar{G}(1, \theta_2).
\]

Define the periodic function \( E : \mathbb{R} \to \mathbb{R} \), with period 2\( \pi \), by

\[
E(y) = \bar{G}(1, y).
\]
From (18) and (19), we see that $E$ is an exponential function, which is periodic with period $2\pi$. Taking $r_2 = 1$, $\theta_1 = 0$ in (15), and applying (14), (17), (19), we obtain

$$G(r_1(\cos \theta_2 + i \sin \theta_2)) = \hat{G}(r_1, 0)\hat{G}(1, \theta_2) = M(r_1)E(\theta_2),$$

and the first part follows.

To solve (9), if $h$ is a constant function solution, then clearly either $h \equiv 0$ or $h \equiv 1$. Assume now that $h$ is a non-constant solution of (9). Replacing $x$ by $x - y/\sqrt{3}$, $y$ by $-2y/\sqrt{3}$, $u$ by $u - v/\sqrt{3}$ and $v$ by $-2v/\sqrt{3}$ into (9), we have

$$h \left( (ux - vy - (uy + vx)/\sqrt{3}, -2(uy + vx)/\sqrt{3} \right) = h \left( x - y/\sqrt{3}, -2y/\sqrt{3} \right) h \left( u - v/\sqrt{3}, -2v/\sqrt{3} \right),$$

(20)

Define the function $H : \mathbb{C} \to \mathbb{R}$ by

$$H(x + iy) = h \left( x - y/\sqrt{3}, -2y/\sqrt{3} \right) \quad (x, y \in \mathbb{R}).$$

(21)

Putting $z = x + iy$, $w = u + iv \in \mathbb{C}$, using (20) and (21), we get

$$H(zw) = H(ux - vy + i(uy + vx)) = h \left( (ux - vy - (uy + vx)/\sqrt{3}, -2(uy + vx)/\sqrt{3} \right)$$

$$= h \left( x - y/\sqrt{3}, -2y/\sqrt{3} \right) h \left( u - v/\sqrt{3}, -2v/\sqrt{3} \right) = H(x + iy)H(u + iv)$$

$$= H(z)H(w),$$

which is of the form of (12). Using similar arguments as in the first part, the second assertion follows. That the function solutions so obtained do indeed satisfy the corresponding functional equations are easily verified.

As a by-product, we now solve (7) using the technique presented in Theorem 1.

**Theorem 2** General solution functions $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation (7) are given by $f \equiv 0$, or $f \equiv 1$, or

$$f(x, y) = M \left( \sqrt{x^2 + y^2} \right) E(\theta_0),$$

where $M$ and $E$ are the multiplicative and exponential functions, $\theta_0$ is the angular coordinate of the polar coordinate of $(y, -x)$.

**Proof.** If $f$ is a constant function solution of (7), then either $f \equiv 0$ or $f \equiv 1$. Assume henceforth that $f$ is a non-constant solution of (7). Substituting $x$ by $-x$ and $u$ by $-u$ into (7), we have

$$f(-uy - vx, vy - ux) = f(-x, y)f(-u, v).$$

(22)

Interchanging $x$ with $y$ and $u$ with $v$, (22) becomes

$$f(-uy - vx, ux - vy) = f(-y, x)f(-v, u).$$

(23)

Putting $F(x, y) := f(-y, x)$, the equation (23) becomes

$$F(ux - vy, uy + vx) = F(x, y)F(u, v).$$

(24)

Defining the function $\tilde{F} : \mathbb{C} \to \mathbb{R}$ via

$$\tilde{F}(x + iy) = F(x, y) \quad (x, y \in \mathbb{R}),$$

(25)

and substituting into (24), we get

$$\tilde{F}(zw) = \tilde{F}(ux - vy + i(uy + vx)) = F(ux - vy, uy + vx) = F(x, y)F(u, v)$$

$$= \tilde{F}(x + iy)\tilde{F}(u + iv) = \tilde{F}(z)\tilde{F}(w),$$

which is of the form of (12). Proceeding in the same manner as in the proof of Theorem 1, the desired result follows. It is easily verified that the functions so obtained satisfy (7).
References


