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Two New Properties Of The Fibonacci Sequence*

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Abstract

In this paper, we establish two new properties of the Fibonacci sequence. The first property deals with an identity based on determinants of certain two by two matrices involving terms of the sequence. The second property is a generalization of a well-known divisibility property.

1 Introduction

One of the best known integer sequences is the Fibonacci sequence, denoted by $\{F_n\}_{n\in\mathbb{N}}$, where $\mathbb{N}$ is the set of non-negative integers $\{0, 1, 2, \cdots\}$. It is an additive sequence defined recursively as

$$F_n = F_{n-1} + F_{n-2}, \tag{1}$$

with the seed values $F_0 = 0$ and $F_1 = 1$. The first few terms of this equation are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots.$$

The sequence can be extended to negative indices, when relevant, by applying a recursion relation based on (1). Indeed, it has been established that the terms with negative indices are related to their positive counterparts as

$$F_{-k} = (-1)^{k+1} F_k \text{ for every } k \in \mathbb{N}.$$

Early discoveries of the Fibonacci sequence are found in the forms of meters in classical Sanskrit poetry dating back to before common era in the South Asia region. Singh [8] mentions that the poems by Āchārya Piṅgala (c. 450 B.C. – 200 B.C.) contain one of the earliest usages of the sequence. The sequence also appears in Nāṭya Śāstra, a Sanskrit treatise in performing arts, composed by Āchārya Bharata (c. 100 B.C. – 350 A.D.). Other scholars in the region, including Virāhaṅka (c. 600 A.D. – 800 A.D.), Gopāla (c. 1135 A.D.) and Hemacandra (c. 1150 A.D.), are also considered to have knowledge of the sequence, see [6, 8]. Later in the thirteenth century in Europe, Fibonacci [2] rediscovered the sequence in the context of describing growth of a rabbit population and published the concept in his book Liber Abaci (1202 A.D.). However, the term *Fibonacci sequence* was first used by Édouard Lucas, a nineteenth century number theorist.

The Fibonacci sequence has remained as one of the objects of central attention to several mathematicians since its introduction. It has been studied in a variety of domains of theories and applications. A review of some modern studies of the sequence can be found in [1, 5], for instance. One of the interesting facts developed about the sequence is its connection to the well-known Golden ratio, often attributed to Johannes Kepler, see [6]. He found that the the sequence $\left\{\frac{F_{n+2}}{F_{n+1}}\right\}_{n\in\mathbb{N}}$ converges to $\frac{1+\sqrt{5}}{2}$, which is the value of the Golden ratio. Furthermore, Binet (1786 A.D. – 1856 A.D.) deduced that the sequence can be expressed explicitly in terms of the Golden ratio, ϕ, as follows.

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}},\ n \in \mathbb{N}. \tag{2}$$

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The expression (2) is commonly known as *Binet's formula* and is generally useful in analytical investigation of the Fibonacci sequence. See [6] for a detailed treatment of the Golden ratio.

Several integer sequences similar to the Fibonacci sequence have been studied in depth and breadth. Some of these include the popular Lucas sequence, Padovan sequence and Narayana's cows sequence, [9, 10]. Similarly, a number of generalizations of the Fibonacci sequences, such as Fibonacci and Lucas polynomials, have been identified and explored extensively over the last few decades, see [7]. While numerous properties and identities related to the Fibonacci sequence have been already discovered and explored to a great extent, the search for undiscovered properties still stands today as an interesting area of investigation in both applied and theoretical mathematics.

In this work, we establish two properties about the Fibonacci sequence, both of which are original to the best of our knowledge. The first property is an identity related to determinants of persymmetric matrices involving terms of the sequence. Its discovery was motivated by a search for properties of determinant structures present in identities such as Cassini's formula and Catalan's identity. We consider a 2×2 matrix determinant defined as

$$A_{mn} = \begin{vmatrix} F_m & F_{n(m+1)} \\ F_{n(m-1)} & F_m \end{vmatrix},$$

for all $m, n \in \mathbb{N}$. Stated next is the first main result of this paper.

Theorem 1 *Let $r, s \in \mathbb{N}$. Then $A_{rs} = A_{sr}$ if and only if r and s have the same parity.*

The second main idea of this paper adds one more result to the large pool of the existing divisibility properties of the Fibonacci sequence. It is known that there are infinitely many terms of the Fibonacci sequence that are divisible by any given integer. However, we want to know how far we need to search in the sequence to find the first few terms that are divisible by a given integer. Given $m, n \in \mathbb{N}$, we define a set of indices of the sequence as

$$D_{m,n} = \{i \in \mathbb{N} : m|F_i \text{ and } i \leq n\}. \tag{3}$$

As the second main result of this paper, we formulate a general statement regarding the cardinality of the set defined in (3) as follows.

Theorem 2 *For $m, r \in \mathbb{N}$, $card(D_{m,2rm}) \geq r$.*

The proof of Theorem 1 is presented in Section 2, and the proof of 2 in Section 3. Some examples and consequences related to these results are discussed in the respective sections.

2 Proof of Theorem 1

The proof of Theorem 1 is inherently based on Catalan's identity. We begin by stating the identity.

Theorem 3 (Catalan's identity) *Let $n, k \in \mathbb{N}$, $n \geq k$. Then*

$$\begin{vmatrix} F_n & F_{n+k} \\ F_{n-k} & F_n \end{vmatrix} = (-1)^{n-k} F_k^2.$$

For a proof of Theorem 3, see [5, p. 106], for instance. Of similar spirit to this identity are Cassini's formula and d'Ocagne's identity, among others. In fact, Cassini's formula is a special case of Catalan's identity with $k = 1$, see [11]. A more general formula can be found in [4].

For the purpose of the proof of our result, we consider a slight modification of Catalan's identity. Replacing n with pq and k with q in Theorem 3 immediately yields us the following.

Lemma 1 *Let $p, q \in \mathbb{N}$. Then*

$$\begin{vmatrix} F_{pq} & F_{q(p+1)} \\ F_{q(p-1)} & F_{pq} \end{vmatrix} = \begin{cases} F_q^2, & q \text{ is even}, \\ (-1)^{p-1} F_q^2, & q \text{ is odd}. \end{cases}$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Suppose $A_{sr} = A_{rs}$. Assume that r and s do not have the same parity. Suppose, without loss of generality, that r is even and s is odd. Let $r = 2m$ and $s = 2n+1$ for some $m, n \in \mathbb{N}$. Then,

$$A_{sr} = \begin{vmatrix} F_{2n+1} & F_{(2m)((2n+1)+1)} \\ F_{(2m)((2n+1)-1)} & F_{2n+1} \end{vmatrix} = F_{2n+1}^2 - F_{(2m)((2n+1)+1)}F_{(2m)((2n+1)-1)}.$$

Replacing p by $2n+1$ and q by $2m$ in Lemma 1 gives

$$-F_{(2m)((2n+1)+1)}F_{(2m)((2n+1)-1)} = F_{2m}^2 - F_{2m(2n+1)}^2.$$

Therefore,

$$A_{sr} = F_{2n+1}^2 + F_{2m}^2 - F_{2m(2n+1)}^2.$$

Similarly, it can be shown that

$$A_{rs} = F_{2m}^2 - F_{2n+1}^2 - F_{2m(2n+1)}^2.$$

Since $A_{sr} = A_{rs}$ by assumption, we have $F_{2n+1}^2 = -F_{2n+1}^2$. This implies $F_{2n+1} = 0$, and hence $2n+1 = 0$, which is not possible.

Conversely, suppose r and s have the same parity. Suppose r and s are both even. Let $r = 2m$ and $s = 2n$ for some $m, n \in \mathbb{N}$. Then,

$$\begin{aligned} A_{sr} &= \begin{vmatrix} F_{2n} & F_{2m(2n+1)} \\ F_{2m(2n-1)} & F_{2n} \end{vmatrix} \\ &= F_{2n}^2 - F_{2m(2n+1)}F_{2m(2n-1)}. \end{aligned}$$

Replacing p by $2n$ and q by $2m$ in Lemma 1 gives

$$-F_{2m(2n+1)}F_{2m(2n-1)} = F_{2m}^2 - F_{(2m)(2n)}^2.$$

Therefore,

$$A_{sr} = F_{2n}^2 + F_{2m}^2 - F_{(2m)(2n)}^2.$$

Similar calculations result in the same expression for A_{rs}. Therefore $A_{sr} = A_{rs}$.

Another case of r and s having the same parity is that they are are both odd. Using similar arguments as above, it can be verified that $A_{sr} = A_{rs}$ in this case as well. ∎

The identity established in Theorem 1 can be restated as follows. Natural numbers $r, s \in \mathbb{N}$ have the same parity if and only if

$$F_r^2 - F_s^2 = F_{s(r+1)}F_{s(r-1)} - F_{r(s+1)}F_{r(s-1)}. \tag{4}$$

Example 1 *Let $r = 102$ and $s = 506$. Then by Theorem 2, we have the equality*

$$\begin{vmatrix} F_{102} & F_{52118} \\ F_{51106} & F_{102} \end{vmatrix} = \begin{vmatrix} F_{506} & F_{51714} \\ F_{51510} & F_{506} \end{vmatrix}.$$

As it can be seen in Example 1, Theorem 2 can be useful in expressing large Fibonacci numbers such as F_{52118} in terms of other ones. Such an expression can play a significant role in analyzing higher terms of the sequence.

Next, we state a consequence of the theorem, which gives a recurrence relation for the even indices of the sequence.

Corollary 1 *Let $k \in \mathbb{N}$ be even. Then*

$$F_k = \begin{cases} 1 & \text{for } k = 2, \\ \frac{F_{k-1}^2 - 1}{F_{k-2}} & \text{for } k > 2. \end{cases}$$

Proof. Letting $r = 1$ and $s = k - 1$ for $k \in \mathbb{N}$ even such that $k > 2$ in (4) gives

$$F_1^2 - F_{k-1}^2 = F_{2(k-1)}F_0 - F_k F_{k-2}.$$

Since $F_0 = 0$ and $F_1 = 1$, the statement follows. ■

Note that Corollary 1 is essentially an instance of Cassini's formula. Other existing well-known identities can also be derived from Theorem 1 upon necessary modifications such as extending it to include negative indices.

A lot of Fibonacci identities have come out elegantly from manipulations of matrices and their determinant properties. Theorem 1 represents a continuation of such a tradition. With further applications and analysis of matrix properties, we believe that additional properties can be derived in a similar manner.

3 Proof of Theorem 2

To prove Theorem 2, we use some known results regarding divisibility within the Fibonacci sequence. We first state the famous strong divisibility property that the sequence obeys.

Theorem 4 *Let $m, n \in \mathbb{N}$. Then $m|n$ if and only if $F_m|F_n$.*

A proof of Theorem 4 can be found in [5], for instance. Next, we state a theorem that deals with the cardinality of the set defined in (3).

Theorem 5 *Let $m \in \mathbb{N}$. Then $card(D_{m,2m}) \geq 1$.*

A proof and a detailed discussion of Theorem 5 can be found in [3], for instance. Now we are ready to prove Theorem 2.

Proof of 2. Let $m, r \in \mathbb{N}$. By Theorem 5, $card(D_{m,2m}) \geq 1$. Suppose $i \in D_{m,2m}$. Then $m|F_i$ and $i \leq 2m$. By Theorem 4, $F_i|F_{ik}$ for all $k \in \mathbb{N}$. Therefore,

$$m|F_{2i},\ m|F_{3i},\ \ldots,\ m|F_{ri}.$$

Since $i \leq 2m$, then $ri \leq 2rm$. Hence,

$$i, 2i, 3i, \ldots, ri \in D_{m,2rm}.$$

That is, $card(D_{m,2rm}) \geq r$. ■

Note that Theorem 2 is a generalization of Theorem 5. An obvious question to ask with respect to Theorem 5 is to find the values of $m \in \mathbb{N}$ such that $card(D_{m,m}) \geq 1$. A quick numerical investigation suggests that the first few $m \in \mathbb{N}$ such that $card(D_{m,m}) \geq 1$ are

$$1, 5, 8, 11, 12, 13, 16, 17, 18, 19, 21, 24, 25, 26, 28, 29, 31, 32, 33, 34, 36, 37, 38, 39, 40 \tag{5}$$

because $1|F_1$, $5|F_5$, $8|F_6$, $11|F_{10}$, $12|F_{12}$ and $13|F_7$. Interestingly, the values in (5) do not seem to be listed in The On-Line Encyclopedia of Integer Sequences, and no obvious pattern appears to be present in the list to the best of our observations.

However, we claim that there are certain classes of $m \in \mathbb{N}$ that divide at least one term of the sequence within the first m terms. In this respect, we have the following conjecture, which has been tested numerically with first 7000 Fibonacci numbers. Its proof eludes us at this time.

Conjecture 1 *Let $k \in \mathbb{N}$ such that $k \leq 50$. Let $M_k = \{kn | k \in \mathbb{N}\}$. For $m \in M_k$, $card(D_{m,m}) \geq 1$ if and only if*

$$k \in \{8, 12, 16, 17, 18, 19, 21, 24, 28, 29, 31, 32, 33, 34, 36, 38, 40, 42, 41, 44, 46, 47, 48\}. \tag{6}$$

Note the the values of k in the conjecture is capped at 50 due to our computational limitations in verifying the conjecture. However, extending it to determine a general list of values of k is an underlying challenge.

The set of values of k in (6) can be contracted to

$$\{8, 12, 17, 18, 19, 21, 28, 29, 31, 33, 41, 44, 46, 47\} \tag{7}$$

since M_{48}, M_{40}, M_{32}, M_{24} and M_{16} are subsets of M_8. Table 1 illustrates Conjecture 1 for the values of k listed in (7) with $n = 1$.

m	8	12	17	18	19	21	28	29	31	33	41	44	46	47
i	6	12	9	12	18	8	24	14	30	20	20	30	24	16

Table 1: Values of $m \in M_k$ with $n = 1$ and $i \in D_{m,m}$

Furthermore, not all the patterns similar to the ones stated in Conjecture 1 satisfy the property. For instance, if $m = 5n$, then for $n = 2$, the smallest Fibonacci number divisible by m is F_{16}.

A further generalization of Conjecture 1 entails finding all $m \in \mathbb{N}$ such that $card(D_{m,m}) \geq \ell$ for a given $\ell \in \mathbb{N}$. Indeed, Conjecture 1 attempts to handle the case of $\ell = 1$ partially. Similar conclusions for the cases $\ell \geq 2$ remain open to explorations.

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