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Orthogonal Polynomials With Respect To A Nonlinear Form*

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Abstract

In this paper, we study properties of the form u satisfying

$$u = -\lambda \left(x^2 - a^2\right)^{-1} v + \delta_0,$$

where v is a regular symmetric semi-classical form (linear functional). We give a necessary and sufficient condition for the regularity of the form u. The coefficients of the three-term recurrence relation, satisfied by the corresponding sequence of orthogonal polynomials, are given explicitly. A study of the semi-classical character of the founded families is done. An example related to the Generalized Gegenbauer form is worked out.

1 Introduction and Preliminaries

The semi-classical forms are a natural generalization of the classical forms (Hermite, Laguerre, Jacobi, and Bessel). Since the system corresponding to the problem of determining all the semi-classical forms of class $s \geq 1$ becomes non-linear, the problem was only solved when $s = 1$ and for some particular cases [2, 5, 16]. Thus, several authors use different processes in order to obtain semi-classical forms of class $s \geq 1$. For instance, let v be a regular form and let us define a new form u by the relation $A(x)u = B(x)v$, where $A(x)$ and $B(x)$ are non-zero polynomials. When $A(x) = 1$, v is positive-definite and $B(x)$ is a positive polynomial, Christoffel [8] has proved that u is still a positive-definite form. This result has been generalized in [9]. The cases $B(x) = \lambda \neq 0$ and $A(x) = x - c$, x^2, x^3, x^4 were treated in [15, 17, 18, 22], where it was shown that under certain regularity conditions the form u is still regular. Moreover, if v is semi-classical, then u is also semi-classical; see also [1, 4, 6, 11, 23, 24, 25]. When $A(x) = B(x)$, u is obtained from v by adding finitely mass points and their derivates [10, 12, 14] and when $A(x)$ and $B(x)$ have no non-trivial common factor, it was found a necessary and sufficient condition for u to be regular in [13]. When $A(x)$ and $B(x)$ are of degree equal to one, an extensive study of the form u has been carried in [27].

In this paper, we consider the situation when $A(x)$ and $B(x)$ are of degree equal to three and one respectively in a particular case. Indeed, we study the form u, fulfilling

$$x\left(x^2 - a^2\right) u = -\lambda x v, \quad (u)_1 = 0, \ (u)_2 = -\lambda \neq 0,$$

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where v is a regular symmetric form. The first section is devoted to the preliminary results and notations used in the sequel. In the second section, an explicit necessary and sufficient condition for the regularity of the new form is given. We obtain the coefficients of the three-term recurrence relation satisfied by the new family of orthogonal polynomials. We also analyze some linear relations linking the polynomials orthogonal with respect to u and v. In the third section, The stability of the semi-classical families is proved. Finally, we apply our result to Generalized Gegenbauer form.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(v)_n := \langle v, x^n\rangle, n \geq 0$, the moments of v. For any form v and any polynomial h let $Dv = v'$, hv, δ_c, and $(x-c)^{-1}v$ be the forms defined by:

$$\langle v', f\rangle := -\langle v, f'\rangle,\ \langle hv, f\rangle := \langle v, hf\rangle,\ \langle \delta_c, f\rangle := f(c),\ \left\langle (x-c)^{-1}v, f\right\rangle := \langle v, \theta_c f\rangle$$

where $(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$, and $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $c, d \in \mathbb{C}$, $c \neq d$, f, $g \in \mathcal{P}$ and $v \in \mathcal{P}'$, we have

$$(x-c)^{-1}\left((x-c)v\right) = v - (v)_0\delta_c, \tag{1}$$

$$(x-c)\left((x-c)^{-1}v\right) = v, \tag{2}$$

$$(x-d)^{-1}\delta_c = \frac{1}{c-d}(\delta_c - \delta_d). \tag{3}$$

cf. [21]. Let us define the operator $\sigma : \mathcal{P} \to \mathcal{P}$ by $(\sigma f)(x) = f(x^2)$. Then, we define the even part σv of v by $\langle \sigma v, f\rangle := \langle v, \sigma f\rangle$. Therefore, we have [20]

$$f(x)(\sigma v) = \sigma\left(f(x^2)v\right), \tag{4}$$

$$\sigma(v') = 2\left(\sigma(xv)\right). \tag{5}$$

A form v is called regular if there exists a sequence of polynomials $\{S_n\}_{n\geq 0}$ ($\deg S_n \leq n$) such that

$$\langle v, S_n S_m\rangle = r_n \delta_{n,m} \ \text{ for } r_n \neq 0 \text{ and } n \geq 0. \tag{6}$$

Then $\deg S_n = n$ for $n \geq 0$ and we can always suppose each S_n is monic. In such a case, the sequence $\{S_n\}_{n\geq 0}$ is unique. It is said to be the sequence of monic orthogonal polynomials with respect to v.

It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [7])

$$\begin{cases} S_{n+2}(x) = \left(x - \xi_{n+1}\right)S_{n+1}(x) - \rho_{n+1}S_n(x) & \text{for } n \geq 0, \\ S_1(x) = x - \xi_0 \text{ and } S_0(x) = 1, \end{cases} \tag{7}$$

with $\left(\xi_n, \rho_{n+1}\right) \in \mathbb{C} \times \mathbb{C} - \{0\}$, $\quad n \geq 0$. By convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first order for the sequence $\{S_n\}_{n\geq 0}$ satisfying the recurrence relation

$$\begin{cases} S_{n+2}^{(1)}(x) = \left(x - \xi_{n+2}\right)S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x) & \text{for } n \geq 0, \\ S_1^{(1)}(x) = x - \xi_1, \ S_0^{(1)}(x) = 1, \text{ and } S_{-1}^{(1)}(x) = 0. \end{cases} \tag{8}$$

Another important representation of $S_n^{(1)}(x)$ is, (see [7]),

$$S_n^{(1)}(x) := \left\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x-\zeta} \right\rangle. \tag{9}$$

Also, let $\{S_n(.,\mu)\}_{n\geq 0}$ be the co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying

$$S_n(x,\mu) = S_n(x) - \mu S_{n-1}^{(1)}(x) \quad \text{for } n \geq 0. \tag{10}$$

cf. [7].

We recall that a form v is called symmetric if $(v)_{2n+1} = 0$ for $n \geq 0$. The conditions $(v)_{2n+1} = 0$ for $n \geq 0$ are equivalent to the fact that the corresponding monic orthogonal polynomial sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (7) with $\xi_n = 0$ for $n \geq 0$. cf. [7].

Throughout this paper, the form v will be supposed normalized, (i.e., $(v)_0 = 1$), symmetric and regular.

Let us consider the decomposition of $\{S_n\}_{n\geq 0}$ and $\{S_n^{(1)}\}_{n\geq 0}$:

$$S_{2n}(x) = P_n(x^2), \quad S_{2n+1}(x) = xR_n(x^2), \tag{11}$$

$$S_{2n}^{(1)}(x) = R_n(x^2, -\rho_1) \text{ and } S_{2n+1}^{(1)}(x) = xP_n^{(1)}(x^2). \tag{12}$$

cf. [7, 20]. The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respective to σv and $x\sigma v$. We also have

$$\begin{cases} R_{n+2}(x) = \left(x - \xi_{n+1}^R\right) R_{n+1}(x) - \rho_{n+1}^R R_n(x) \quad \text{for } n \geq 0, \\ R_1(x) = x - \xi_0^R \quad \text{and} \quad R_0(x) = 1, \end{cases} \tag{13}$$

with

$$\xi_0^R = \rho_1 + \rho_2,\ \xi_{n+1}^R = \rho_{2n+3} + \rho_{2n+4}, \text{ and } \rho_{n+1}^R = \rho_{2n+2}\rho_{2n+3} \text{ for } n \geq 0. \tag{14}$$

By virtue of (8), with $\xi_n = 0$, we get $S_{n+2}^{(1)}(0) = -\rho_{n+2} S_n^{(1)}(0)$. Consequently,

$$S_{2n}^{(1)}(0) = R_n(0, -\rho_1) = (-1)^n \prod_{\nu=0}^{n} \rho_{2\nu} \text{ for } n \geq 0. \tag{15}$$

PROPOSITION 1 ([7, 21]). v is regular if and only if σv and $x\sigma v$ are regular.

2 Algebraic Properties

For fixed $a \in \mathbb{C}$ and $\lambda \in \mathbb{C} - \{0\}$, we can define a new normalized form $u \in \mathcal{P}'$ by the relation

$$u = -\lambda(x^2 - a^2)^{-1} v + \delta_0. \tag{16}$$

Equivalently, from (1)-(3) we have

$$x\left(x^2-a^2\right)u=-\lambda xv,\ (u)_1=0,\ \text{and}\ \ (u)_2=-\lambda. \tag{17}$$

The case $a=0$ is treated in [1, 18, 26], so henceforth, we assume $a\neq 0$.

PROPOSITION 2. u is regular if and only if

$$R_n(a^2,-\rho_1)\Delta_n\neq 0\ \text{ for } n\geq 0, \tag{18}$$

where R_n is defined by (13), and for $n\geq 0$,

$$\begin{aligned}\Delta_n=R_{n+1}(a^2,-\rho_1)&\left(\lambda R_n(0,-\rho_1)+a^2R_n(0)\right)\\ -R_n(a^2,-\rho_1)&\left(\lambda R_{n+1}(0,-\rho_1)+a^2R_{n+1}(0)\right).\end{aligned} \tag{19}$$

PROOF. Multiplying (17) by x and applying the operator σ for the obtained equation and using (2), we get

$$-\lambda^{-1}x\sigma u=\rho_1\left(x-a^2\right)^{-1}\left(\rho_1^{-1}x\sigma v\right)+\delta_{a^2}. \tag{20}$$

From (20) and (3), we get

$$\sigma u=-\lambda\rho_1x^{-1}\left(x-a^2\right)^{-1}\left(\rho_1^{-1}x\sigma v\right)+\left(1+\frac{\lambda}{a^2}\right)\delta_0-\frac{\lambda}{a^2}\delta_{a^2}. \tag{21}$$

From (16), it is plain that u is a symmetric form. Then, according to Proposition 1, u is regular if and only if $x\sigma u$ and σu are regular. But

$$-\lambda^{-1}x\sigma u=\rho_1\left(x-a^2\right)^{-1}\left(\rho_1^{-1}x\sigma v\right)+\delta_{a^2}$$

is regular if and only if $\lambda\neq 0$ and $R_n(a^2,-\rho_1)\neq 0$ for $n\geq 0$ (see [22]). So u is regular if and only if $R_n(a^2,-\rho_1)\neq 0$ and

$$\sigma u=-\lambda\rho_1x^{-1}\left(x-a^2\right)^{-1}\left(\rho_1^{-1}x\sigma v\right)+\left(1+\frac{\lambda}{a^2}\right)\delta_0-\frac{\lambda}{a^2}\delta_{a^2}$$

is regular. Or, it was shown in [6] that the form

$$-\lambda\rho_1x^{-1}\left(x-a^2\right)^{-1}\left(\rho_1^{-1}x\sigma v\right)+\left(1+\frac{\lambda}{a^2}\right)\delta_0-\frac{\lambda}{a^2}\delta_{a^2}$$

is regular if and only if $\Delta_n\neq 0$ for $n\geq 0$. Then, we deduce the desired result.

REMARK 1. From (11) and (12), we get

$$R_n(a^2,-\rho_1)=S_{2n}^{(1)}(a),\quad R_n(0,-\rho_1)=S_{2n}^{(1)}(0),\ \ \text{and}\ \ R_n(0)=S'_{2n+1}(0)$$

for $n \geq 0$. Thus, u is regular if and only if

$$S_{2n}^{(1)}(a)\Big\{S_{2n+2}^{(1)}(a)\left(\lambda S_{2n}^{(1)}(0)+a^2S'_{2n+1}(0)\right) - S_{2n}^{(1)}(a)\left(\lambda S_{2n+2}^{(1)}(0)+a^2S'_{2n+3}(0)\right)\Big\} \neq 0 \quad \text{for} \quad n \geq 0. \tag{22}$$

REMARK 2. From (7), we have

$$S'_1(0)=1 \quad \text{and} \quad S'_{2n+3}(0)=S_{2n+2}(0)-\rho_{2n+3}S'_{2n+1}(0) \quad \text{for } n \geq 0.$$

Therefore, we can easily prove by induction that

$$S'_{2n+1}(0)=(-1)^n\Lambda_n S_{2n}^{(1)}(0) \quad \text{for } n \geq 0, \tag{23}$$

with

$$\Lambda_n = 1+\sum_{\nu=0}^{n-1}\prod_{k=0}^{\nu}\frac{\rho_{2k+1}}{\rho_{2k+2}} \quad \text{for } n \geq 0 \text{ where } \sum_{\nu=0}^{-1}=0. \tag{24}$$

When u is regular, let $\{Z_n\}_{n\geq 0}$ be the corresponding sequence satisfying the recurrence relation

$$\begin{cases} Z_{n+2}(x)=xZ_{n+1}(x)-\gamma_{n+1}Z_n(x) \quad \text{for } n \geq 0, \\ Z_1(x)=x \quad \text{and } Z_0(x)=1. \end{cases} \tag{25}$$

Let us now consider the quadratic decomposition of the sequence $\{Z_n\}_{n\geq 0}$

$$Z_{2n}(x)=\tilde{P}_n(x^2) \quad \text{and} \quad Z_{2n+1}(x)=x\tilde{R}_n(x^2) \quad \text{for } n \geq 0. \tag{26}$$

From (20) and (21), we can deduce the following results.

PROPOSITION 3 ([22]). The polynomials of the sequence $\{\tilde{R}_n\}_{n\geq 0}$ satisfy the relation

$$\tilde{R}_{n+1}(x)=R_{n+1}(x)+a_nR_n(x) \quad \text{for } n \geq 0, \tag{27}$$

where

$$a_n=-\frac{S_{2n+2}^{(1)}(a)}{S_{2n}^{(1)}(a)} \quad \text{for } n \geq 0. \tag{28}$$

PROPOSITION 4 ([6]). The polynomials of the sequence $\{\tilde{P}_n\}_{n\geq 0}$ satisfy the relation

$$\begin{cases} \tilde{P}_{n+2}(x)=R_{n+2}(x)+c_{n+1}R_{n+1}(x)+b_nR_n(x) \quad \text{for } n \geq 0, \\ \tilde{P}_1(x)=R_1(x)+c_0, \end{cases} \tag{29}$$

where

$$b_n = -\frac{\Delta_{n+1}}{\Delta_n} \quad \text{for } n \geq 0, \tag{30}$$

and, for $n \geq 0$,

$$\begin{cases} c_{n+1} = -\Delta_n^{-1}\Big\{S_{2n}^{(1)}(a)\left(\lambda S_{2n+2}^{(1)}(0) + a^2 S'_{2n+5}(0)\right) \\ \qquad\qquad -S_{2n+4}^{(1)}(a)\left(\lambda S_{2n}^{(1)}(0) + a^2 S'_{2n+1}(0)\right)\Big\}, \\ c_0 = -\lambda - \rho_1 - \rho_2. \end{cases} \tag{31}$$

LEMMA 1.

$$\begin{aligned} &xZ_{n+3}(x) = S_{n+4}(x) + \tilde{b}_{n+2}S_{n+2}(x) + \tilde{a}_n S_n(x) \quad \text{for } n \geq 0, \\ &xZ_2(x) = S_3(x) + \tilde{b}_1 S_1(x), \\ &xZ_1(x) = S_2(x) + \tilde{b}_0, \end{aligned} \tag{32}$$

with for $n \geq 0$,

$$\begin{aligned} &\tilde{a}_{2n} = \rho_{2n+1}a_n, \quad \tilde{a}_{2n+1} = b_n, \\ &\tilde{b}_{2n+2} = \rho_{2n+3} + a_n, \quad \tilde{b}_{2n+3} = c_{n+1}, \\ &\tilde{b}_0 = \rho_1 \quad \text{and} \quad \tilde{b}_1 = c_0. \end{aligned} \tag{33}$$

PROOF. From (26),we have

$$xZ_{2n+2}(x) = x\tilde{P}_{n+1}(x^2) \quad \text{and} \quad xZ_{2n+1}(x) = x^2\tilde{R}_n(x^2) \quad \text{for } n \geq 0.$$

Then, from the above equation, (11), (27) and (29), we get (32).

PROPOSITION 5. We may write

$$\gamma_1 = -\lambda, \quad \gamma_{n+2} = \rho_{n+1}\frac{\tilde{a}_{n+1}}{\tilde{a}_n}, \tag{34}$$

$$\gamma_{n+3} - \rho_{n+3} = \tilde{b}_{n+2} - \tilde{b}_{n+3}, \tag{35}$$

and

$$\tilde{a}_{n+1} - \tilde{a}_n = \rho_{n+2}\tilde{b}_{n+2} - \gamma_{n+3}\tilde{b}_{n+1}, \tag{36}$$

for $n \geq 0$.

PROOF. After multiplication of (32) by x, we apply the recurrence relations (7) and (25), we get

$$\begin{aligned} xZ_{n+4}(x) + \gamma_{n+3}xZ_{n+2}(x) \quad &= \quad S_{n+5}(x) + (\rho_{n+4} + \tilde{b}_{n+2})S_{n+3}(x) \\ &\quad +(\tilde{a}_n + \rho_{n+2}\tilde{b}_{n+2})S_{n+1}(x) + \rho_n\tilde{a}_n S_{n-1}(x) \end{aligned}$$

for $n \geq 1$. Substituting xZ_{k+3} in the above equation by $S_{k+4} + \tilde{b}_{k+2}S_{k+2} + \tilde{a}_k S_k$ with $k = n+1,\ n-1$, we obtain (34)-(36), after comparing the coefficients of S_k with $k = n+3,\ n+1,\ n-1$.

REMARK 3. From (14), (33) and (34), the sequence $\{\tilde{R}_n\}_{n\geq 0}$ satisfies the recurrence relation (13) with for $n \geq 0$,

$$\beta_0^{\tilde{R}} = -\lambda - \frac{b_0}{a_0}, \quad \beta_{n+1}^{\tilde{R}} = \rho_{2n+2}\rho_{2n+3}\frac{a_{n+1}}{b_n} + \frac{b_{n+1}}{a_{n+1}},$$

and

$$\gamma_{n+1}^{\tilde{R}} = \rho_{2n+2}\rho_{2n+3}\frac{a_{n+1}}{a_n}.$$

3 The Semi-Classical Case

In this section, we compute the exact class of the semi-classical form u.

DEFINITION 1 ([21]). The form v is called semi-classical when it is regular and satisfies the Riccati equation

$$\Phi(z)S'\big(v\big)(z) = C(z)S\big(v\big)(z) + D(z), \tag{37}$$

where Φ monic, C and D are polynomials and $S(v)(z)$ designes the formal Stieltjes function of the form v defined by:

$$S(v)(z) = -\sum_{n\geq 0}\frac{(v)_n}{z^{n+1}}. \tag{38}$$

It was shown in [21] that equation (37) is equivalent to

$$(\Phi(x)v)' + \Psi v = 0, \tag{39}$$

with

$$\Psi(x) = -\Phi'(x) - C(x). \tag{40}$$

We also have the following relation :

$$D(x) = -\left(v\theta_0\Phi\right)'(x) - \left(v\theta_0\Psi\right)(x).$$

PROPOSITION 6 ([21]). Define $r = \deg(\Phi)$ and $p = \deg(\Psi)$. The semi-classical form v satisfying (39) is of class $s = \max\left(r-2, p-1\right)$ if and only if

$$\prod_{c\in\mathcal{Z}}\left\{|\Phi'(c) + \Psi(c)| + \left|\langle v, \theta_c^2\Phi + \theta_c\Psi\rangle\right|\right\} \neq 0, \tag{41}$$

where $\mathcal{Z}$ denotes the set of zeros of Φ.

COROLLARY 1 ([19]). The form v satisfying (37) is of class s if and only if

$$\prod_{c\in\mathcal{Z}} \left(|C(c)| + |D(c)|\right) \neq 0. \tag{42}$$

PROPOSITION 7. If v is a semi-classical form and satisfies (37), then for every $\lambda \in \mathbb{C} - \{0\}$ such that $R_n(a^2, -\rho_1)\Delta_n \neq 0, n \geq 0$, the form u defined by (16) is regular and semi-classical. It satisfies

$$\tilde{\Phi}(z)S'(u)(z) = \tilde{C}(z)S(u)(z) + \tilde{D}(z), \tag{43}$$

where

$$\begin{cases} \tilde{\Phi}(z) = z^2\left(z^2 - a^2\right)\Phi(z), \\ \tilde{C}(z) = z^2\left(z^2 - a^2\right)C(z) - 2z^3\Phi(z), \\ \tilde{D}(z) = z\left(z^2 - a^2\right)C(z) - \left(z^2 + a^2\right)\Phi(z) - \lambda z^2 D(z), \end{cases} \tag{44}$$

and u is of class $\tilde{s}$ such that $\tilde{s} \leq s + 4$.

PROOF. We have [21]

$$zS(v)(z) = S(\xi v)(z) - \left(v\theta_0(\xi)\right)(z) = S(\xi v)(z) - 1.$$

Using (17), we get

$$zS(v)(z) = -\frac{1}{\lambda}S\left(\xi\left(\xi - a\right)\left(\xi - b\right)u\right)(z) - 1 = -\frac{1}{\lambda}\left(z - a\right)\left(z - b\right)\left(zS(u)(z) - 1\right). \tag{45}$$

Multiplying (37) by z^2 and taking into account (45) we obtain (43)-(44).

From (39) and (43)-(44), the form u satisfies the distributional equation

$$\left(\tilde{\Phi}(x)v\right)' + \tilde{\Psi}v = 0, \tag{46}$$

where $\tilde{\Phi}$ is the polynomial defined in (44) and

$$\tilde{\Psi}(x) = -\tilde{\Phi}'(x) - \tilde{C}(x) = x\left(x^2 - a^2\right)\left(x\Psi(x) - 2\Phi(x)\right). \tag{47}$$

Then $\deg(\tilde{\Phi}) = \tilde{r} \leq s + 6$ and $\deg(\tilde{\Psi}) = \tilde{p} \leq s + 5$. Thus $\tilde{s} = \max\left(\tilde{r} - 2, \tilde{p} - 1\right) \leq s + 4$.

PROPOSITION 8. Let u be a semi-classical form satisfying (43). For every zero of $\tilde{\Phi}$ different from 0 and a, the equation (43) is irreducible.

PROOF. Since v is a semi-classical form of class s, $S(v)(z)$ satisfies (37), where the polynomials Φ, C and D are coprime. Let $\tilde{\Phi}$, $\tilde{C}$ and $\tilde{D}$ be as in Proposition 7. Let c be a zero of $\tilde{\Phi}$ different from 0 and a, this implies that $\Phi(c) = 0$.

We know that $|C(c)| + |D(c)| \neq 0$, (i) if $C(c) \neq 0$, then $\tilde{C}(c) \neq 0$; and (ii) if $C(c) = 0$ and $D(c) \neq 0$, then $\tilde{D}(c) \neq 0$, whence $|\tilde{C}(c)| + |\tilde{D}(c)| \neq 0$.

Concerning the class of u, we have the following result (see Proposition 9). But first, let us recall this technical lemma.

LEMMA 2. We have the following properties:

(P_1) The equation (43)-(44) is irreducible in 0 if and only if

$$\Phi(0) \neq 0.$$

(P_2) The equation (43)-(44) is divisible by z but not by z^2 if and only if

$$\Phi(0) = 0 \ \text{ and } \ C(0) + \Phi'(0) \neq 0.$$

(P_3) The equation (43)-(44) is irreducible in a and $-a$ if and only if

$$|\Phi(a)| + |D(a)| \neq 0.$$

(P_4) The equation (43)-(44) is divisible by $z^2 - a^2$ but not by $(z^2 - a^2)^2$ if and only if

$$\Phi(a) = D(a) = 0 \ \text{ and } \ |C(a) - a\Phi''(a)| + |D''(a)| \neq 0.$$

PROOF. From (44), we have $\tilde{C}(0) = 0$ and $\tilde{D}(0) = -a^2\Phi(0)$. If $\Phi(0) \neq 0$, then $\tilde{D}(0) \neq 0$. So, by virtue of (42), we obtain P_1. Now, if $\Phi(0) = 0$, then the equation (43)-(44) is divisible by z according to (42). Thus $S(u)(z)$ satisfies (43) with

$$\begin{cases} \tilde{\Phi}(z) = z\left(z^2 - a^2\right)\Phi(z), \\ \tilde{C}(z) = z\left(z^2 - a^2\right)C(z) - 2z^2\Phi(z), \\ \tilde{D}(z) = \left(z^2 - a^2\right)C(z) - \left(z^2 + a^2\right)\left(\theta_0\Phi\right)(z) - \lambda z D(z). \end{cases} \tag{48}$$

Therefore, $\tilde{C}(0) = 0$ and $\tilde{D}(0) = -a^2\left(C(0) + \Phi'(0)\right)$. If $C_0(0) + \Phi'(0) \neq 0$, then the equation (43)-(48) is irreducible in 0. Thus, we deduce P_2. From (44), we get $\tilde{C}(a) = -2a^3\Phi(a)$ and $\tilde{D}(a) = -a^2\left(\lambda D(a) + 2\Phi(a)\right)$. Then, we can deduce that $|\tilde{C}(a)| + |\tilde{D}(a)| \neq 0$ if and only if $(\Phi(a), D(a)) \neq (0,0)$. Thus P_3 is proved. If $(\Phi(a), D(a)) = (0,0)$, then the equation (43)-(44) can be divided by $z^2 - a^2$ since u is symmetric and according to (42). In this case $S(u)(z)$ satisfies (43) with

$$\begin{cases} \tilde{\Phi}(z) = z^2\Phi(z), \\ \tilde{C}(z) = z^2C(z) - 2z^3\left(\theta_{-a}\theta_a\Phi\right)(z), \\ \tilde{D}(z) = zC(z) - \left(z^2 + a^2\right)\left(\theta_{-a}\theta_a\Phi\right)(z) - \lambda z^2\left(\theta_{-a}\theta_a D\right)(z). \end{cases} \tag{49}$$

Substituting z by a in (49), we obtain

$$\tilde{C}(a) = a^2\left(C(a) - a\Phi''(a)\right), \quad \tilde{D}(z) = a\left(C(a) - a\Phi''(a) - \lambda\frac{a}{2}D''(a)\right).$$

Then (43)-(49) is irreducible in a and $-a$ if and only if $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$. Hence P_4.

PROPOSITION 9. Under the conditions of Proposition 7, for the class of u, we have two different cases:

(A) $\Phi(0) \neq 0$

(i) $\tilde{s} = s + 4$ if $(\Phi(a), D(a)) \neq (0, 0)$.
(ii) $\tilde{s} = s + 2$ if $(\Phi(a), D(a)) = (0, 0)$ and $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$.

(B) $\Phi(0) = 0$ and $C(0) + \Phi'(0) \neq 0$

(i) $\tilde{s} = s + 3$ if $(\Phi(a), D(a)) \neq (0, 0)$.
(ii) $\tilde{s} = s + 1$ if $(\Phi(a), D(a)) = (0, 0)$ and $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$.

PROOF. From Proposition 8, the class of u depends only on the zeros 0 and a. For the zero 0 we consider the following situation:

(A) $\Phi(0) \neq 0$. In this case the equation (43)-(44) is irreducible in 0 according to P_1. But what about the zero a? We will analyze the following cases:

(i) $(\Phi(a), D(a)) \neq (0, 0)$, the equation (43)-(44) is irreducible in a and $-a$ according to P_3. Then $\tilde{s} = s + 4$. Thus we proved (A)(i).

(ii) $(\Phi(a), D(a)) = (0, 0)$ and $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$.

From P_3 and P_4, (43)-(44) is divisible by $z^2 - a^2$ but not by $\left(z^2 - a^2\right)^2$ and thus the order of the class of u decreases in two units. In fact, $S(u)(z)$ satisfies the irreducible equation (43)-(49) and then $\tilde{s} = s + 2$. Hence (A)(ii).

(B) $\Phi(0) = 0$ and $C(0) + \Phi'(0) \neq 0$.

In this condition, (43)-(44) is divisible by z but not by z^2 according to P_2. But what about the zero a? We have the two following cases:

(i) $(\Phi(a), D(a)) \neq (0, 0)$, the equation (43)-(44) is irreducible in a and $-a$ according to P_3. Therefore $S(u)(z)$ satisfies the irreducible equation (43)-(49) and then $\tilde{s} = s + 3$ and (B)(i) is also proved.

(ii) $(\Phi(a), D(a)) = (0, 0)$ and $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$.

From P_3 and P_4, (43)-(48) is divisible by $z^2 - a^2$ but not by $\left(z^2 - a^2\right)^2$. Then, $S(u)(z)$ satisfies the irreducible equation (43) with

$$\begin{cases} \tilde{\Phi}(z) = z\Phi(z), \\ \tilde{C}(z) = zC(z) - 2z^2\left(\theta_{-a}\theta_a\Phi\right)(z), \\ \tilde{D}(z) = C(z) - \left(z^2 + a^2\right)\left(\theta_0\theta_{-a}\theta_a\Phi\right)(z) - \lambda z\left(\theta_{-a}\theta_a D\right)(z), \end{cases} \tag{50}$$

and thus $\tilde{s} = s + 1$.

Finally, if we suppose that the form v has the following integral representation:

$$\langle v,\, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx \quad \text{for } f \in \mathcal{P} \text{ with } (v)_0 = \int_{-\infty}^{+\infty} V(x)dx = 1,$$

where V is locally integrable function with rapid decay and continuous at a and $-a$. Then, from (16) the form u is represented by

$$\begin{aligned} \langle u,\, f\rangle = f(0) + \tfrac{\lambda}{2a}\Big\{ & P\textstyle\int_{-\infty}^{+\infty} \frac{V(x)}{x+a} f(x)dx - P\int_{-\infty}^{+\infty} \frac{V(x)}{x-a} f(x)dx \\ & + \Big(f(a) + f(-a)\Big) P\textstyle\int_{-\infty}^{+\infty} \frac{V(x)}{x-a}dx \Big\}, \end{aligned} \tag{51}$$

where for $c \in \{a, -a\}$

$$P\int_{-\infty}^{+\infty} \frac{V(x)}{x-c} f(x)dx = \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{c-\varepsilon} \frac{V(x)}{x-c} f(x)dx + \int_{c+\varepsilon}^{+\infty} \frac{V(x)}{x-c} f(x)dx \right].$$

4 Application

Proposition 7 shows that we can generate new semi-classical sequences from well known ones. We apply our results to $v := G.G(\alpha,\beta)$, where $G.G(\alpha,\beta)$ is the Generalized Gegenbauer form. In this case, the form v is symmetric semi-classical of class $s = 1$. Thus, we have [7]

$$\begin{cases} \rho_{2n+1} = \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} & \text{for } n \geq 0, \\ \rho_{2n+2} = \frac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)} & \text{for } n \geq 0. \end{cases} \tag{52}$$

The regularity conditions are $\alpha \neq -n$, $\beta \neq -n$, $\alpha+\beta \neq -n$, $n \geq 1$. We also have

$$\begin{array}{l} \Phi(x) = x\left(x^2-1\right) \ , \quad \Psi(x) = -2\left(\alpha+\beta+2\right)x^2 + 2\left(\beta+1\right), \\ C(x) = \left(2\alpha+2\beta+1\right)x^2 - \left(2\beta+1\right) \ , \quad D(x) = 2\left(\alpha+\beta+1\right)x. \end{array} \tag{53}$$

For greater convenience we take $a = 1$, and $\alpha \neq 0$. From (8) and (52), we can easily obtain by induction

$$S_{2n}^{(1)}(0) = (-1)^n \frac{\Gamma\left(\alpha+\beta+2\right)\Gamma\left(n+\alpha+1\right)}{\Gamma\left(\alpha+1\right)\Gamma\left(2n+\alpha+\beta+2\right)} \quad \text{for } n \geq 0, \tag{54}$$

and

$$S_{2n}^{(1)}(1) = \frac{\alpha+\beta+1}{\alpha\Gamma\left(2n+\alpha+\beta+2\right)}\Omega_n \quad \text{for } n \geq 0, \tag{55}$$

with, for $n \geq 0$,

$$\Omega_n = \frac{\Gamma\left(n+\alpha+1\right)\Gamma\left(n+\alpha+\beta+2\right)}{\Gamma\left(\alpha+1\right)} - \frac{\Gamma\left(\alpha+\beta+1\right)\Gamma\left(n+1\right)\Gamma\left(n+\beta+2\right)}{\Gamma\left(\beta+1\right)}.$$

From (52), we get

$$\frac{\rho_{2k+1}}{\rho_{2k+2}} = \frac{\left(k+\beta+1\right)\left(k+\alpha+\beta+1\right)\left(2k+\alpha+\beta+3\right)}{\left(k+1\right)\left(k+\alpha+1\right)\left(2k+\alpha+\beta+1\right)}.$$

Then

$$\begin{aligned} \prod_{k=0}^{\nu} \frac{\rho_{2k+1}}{\rho_{2k+2}} &= \left(2\nu+\alpha+\beta+3\right)\frac{\Gamma\left(\alpha+1\right)\Gamma\left(\nu+\beta+2\right)\Gamma\left(\nu+\alpha+\beta+2\right)}{\Gamma\left(\beta+1\right)\Gamma\left(\alpha+\beta+2\right)\Gamma\left(\nu+2\right)\Gamma\left(\nu+\alpha+2\right)} \\ &= \frac{\left(2\nu+\alpha+\beta+3\right)\Gamma\left(\alpha+1\right)}{\Gamma\left(\beta+1\right)\Gamma\left(\alpha+\beta+2\right)\left(\nu+1\right)\left(\nu+\alpha+1\right)}h_\nu, \end{aligned}$$

with

$$h_n = \frac{\Gamma(n+\beta+2)\,\Gamma(n+\alpha+\beta+2)}{\Gamma(n+1)\,\Gamma(n+\alpha+1)}\ ,\ n \geq 0,$$

fulfilling

$$h_{n+1} = \frac{(n+\beta+2)\,(n+\alpha+\beta+2)}{(n+1)\,(n+\alpha+1)} h_n\ ,\ n \geq 0.$$

Therefore

$$h_{n+1} - h_n = \frac{(\beta+1)\,(2n+\alpha+\beta+3)}{(n+1)\,(n+\alpha+1)} h_n\ ,\ n \geq 0,$$

and consequently, from the above results, we obtain that for $n \geq 1$,

$$\begin{aligned}\sum_{\nu=0}^{n-1}\prod_{k=0}^{\nu}\frac{\rho_{2k+1}}{\rho_{2k+2}} &= \frac{\Gamma(\alpha+1)}{\Gamma(\beta+2)\,\Gamma(\alpha+\beta+2)}\sum_{\nu=0}^{n-1}(h_{\nu+1}-h_\nu) \\ &= \frac{\Gamma(\alpha+1)\,\Gamma(n+\beta+2)\,\Gamma(n+\alpha+\beta+2)}{\Gamma(\beta+2)\,\Gamma(\alpha+\beta+2)\,\Gamma(n+1)\,\Gamma(n+\alpha+1)} - 1.\end{aligned}$$

Finally, (24), (23) and (19) become respectively

$$\Lambda_n = \frac{\Gamma(\alpha+1)\,\Gamma(n+\beta+2)\,\Gamma(n+\alpha+\beta+2)}{\Gamma(\beta+2)\,\Gamma(\alpha+\beta+2)\,\Gamma(n+1)\,\Gamma(n+\alpha+1)} \quad \text{for } n \geq 0, \tag{56}$$

$$S'_{2n+1}(0) = \frac{\Gamma(n+\beta+2)\,\Gamma(n+\alpha+\beta+2)}{\Gamma(\beta+2)\,\Gamma(n+1)\,\Gamma(2n+\alpha+\beta+2)} \quad \text{for } n \geq 0, \tag{57}$$

and

$$\Delta_n = \frac{\alpha+\beta+1}{\alpha\Gamma(2n+\alpha+\beta+2)\,\Gamma(2n+\alpha+\beta+3)}\left(\Theta_n\lambda + \Upsilon_n\right) \quad \text{for } n \geq 0, \tag{58}$$

with for $n \geq 0$

$$\Theta_n = (-1)^n \frac{\Gamma(\alpha+\beta+2)\,\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\left(\Omega_{n+1} + (n+\alpha+1)\,\Omega_n\right),$$

$$\begin{aligned}\Upsilon_n &= \frac{\Gamma(n+\beta+2)\,\Gamma(n+\alpha+\beta+2)}{\Gamma(\beta+2)\,\Gamma(n+2)}[(n+1)\,\Omega_{n+1} \\ &\quad + (n+\beta+2)\,(n+\alpha+\beta+2)\,\Omega_n].\end{aligned}$$

Thus, u is regular for every $\lambda \neq 0$ such that

$$\Omega_n(\Theta_n\lambda + \Upsilon_n) \neq 0 \quad \text{for } n \geq 0.$$

Using (55) and (58), we obtain for (28) and (30) (for $n \geq 0$)

$$a_n = -\frac{\Omega_{n+1}}{\Omega_n\,(2n+\alpha+\beta+2)\,(2n+\alpha+\beta+3)},$$

$$b_n = -\frac{\Theta_{n+1}\lambda + \Upsilon_{n+1}}{(\Theta_n\lambda + \Upsilon_n)(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)(2n+\alpha+\beta+5)}.$$

Therefore, we have for (34)

$$\gamma_1 = -\lambda,$$
$$\gamma_{2n+2} = \frac{\Omega_n(\Theta_{n+1}\lambda + \Upsilon_{n+1})}{\Omega_{n+1}(\Theta_n\lambda + \Upsilon_n)(2n+\alpha+\beta+4)(2n+\alpha+\beta+5)},$$
$$\gamma_{2n+3} = \frac{\Omega_{n+2}(\Theta_n\lambda + \Upsilon_n)(n+1)(n+\alpha+1)(n+\beta+2)(n+\alpha+\beta+2)}{\Omega_{n+1}(\Theta_{n+1}\lambda + \Upsilon_{n+1})(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)}.$$

Since v is semi-classical, then according to Proposition 7, (40) and (48), the form u is also semi-classical of class $\tilde{s} = 4$ and fulfils (43) and (47) with

$$\tilde{\Phi}(x) = x^2\left(x^2-1\right)^2,$$
$$\tilde{\Psi}(x) = -x\left(x^2-1\right)\left((2\alpha+2\beta+5)x^2 - 2\beta - 3\right),$$
$$\tilde{C}(x) = x\left(x^2-1\right)\left((2\alpha+2\beta-1)x^2 - 2\beta - 1\right),$$
$$\tilde{D}(x) = 2(\beta+1)x^4 - 2((\alpha+\beta+1)(\lambda+1)+\beta)x^2 + 2(\beta+1).$$

The form v has the following integral representation [7], for $\Re\alpha > -1$, $\Re\beta > -1$, $f \in \mathcal{P}$,

$$\langle v, f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}|x|^{2\beta+1}\left(1-x^2\right)^{\alpha}f(x)dx.$$

Then, from (51), we obtain

$$\begin{aligned}\langle u, f\rangle &= f(0) + \lambda\frac{\Gamma(\alpha+\beta+2)}{2\Gamma(\alpha+1)\Gamma(\beta+1)}\left[\int_{-1}^{1}\frac{|x|^{2\beta+1}\left(1-x^2\right)^{\alpha}}{x+1}(f(x)-f(-1))dx\right.\\ &\left.-\int_{-1}^{1}\frac{|x|^{2\beta+1}\left(1-x^2\right)^{\alpha}}{x-1}(f(x)-f(1))dx\right],\end{aligned}$$

for $\Re\alpha > -1$ and $\Re\beta > -1$. But, if $\Re\alpha > 0$, we have

$$\int_{-1}^{1}\frac{|x|^{2\beta+1}\left(1-x^2\right)^{\alpha}}{x+1}dx = -\int_{-1}^{1}\frac{|x|^{2\beta+1}\left(1-x^2\right)^{\alpha}}{x-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

Consequently, if $\Re\alpha > 0$, $\Re\beta > -1$, $f \in \mathcal{P}$,

$$\begin{aligned}\langle u, f\rangle &= \lambda\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}|x|^{2\beta+1}\left(1-x^2\right)^{\alpha-1}f(x)dx\\ &+f(0) - \lambda\frac{\alpha+\beta+1}{2\alpha}(f(1)+f(-1)).\end{aligned} \tag{59}$$

REMARKS 4. From (59),we have

$$u = \lambda\frac{\alpha+\beta+1}{\alpha}G.G(\alpha-1,\beta) + \delta_0 - \lambda\frac{\alpha+\beta+1}{2\alpha}(\delta_1 + \delta_{-1}).$$

For more details see [3]. Using (59), we get

$$(u)_{2n+2} = \lambda \frac{\Gamma(\alpha+\beta+2)\,\Gamma(n+\beta+2)}{\alpha\Gamma(\beta+1)\,\Gamma(n+\alpha+\beta+2)} - \lambda \frac{\alpha+\beta+1}{\alpha}\ ,\ n \geq 0.$$

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