Applied Mathematics E-Notes, 14(2014), 116-122 © ISSN 1607-2510
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Some L_r Inequalities Involving The Polar Derivative Of A Polynomial*

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Received 2 May 2014

Abstract

Let $P(z)$ be a polynomial of degree n and for $\alpha \in \mathbb{C}$, let $D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of the polynomial $P(z)$ with respect to α. In this paper, we obtain L_r mean extension of some inequalities concerning the polar derivative of a polynomial having all zeros inside a circle. Our results generalize and sharpen some well-known polynomial inequalities.

1 Introduction

Let $P(z)$ be a polynomial of degree n, then concerning the estimate for the upper bound of the maximum modulus of $|P'(z)|$ in terms of the maximum modulus of $|P(z)|$ on the unit circle $|z| = 1$, we have

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)| .$$

Inequality (1) is a famous result known as Bernstein's Inequality (for reference see [10]). Equality in (1) holds if and only if $P(z)$ has all its zeros at the origin. For the polynomials having all their zeros in the disk $|z| \le 1$, Paul Turán [13] estimated the lower bound for the maximum modulus of $|P'(z)|$ on $|z| = 1$ by showing that if $P(z)$ is a polynomial of degree n and has all its zeros in $|z| \le 1$, then

$$n \max_{|z|=1} |P(z)| \le 2 \max_{|z|=1} |P'(z)| . \tag{1}$$

Inequality (1) is best possible with equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta| \neq 0$.

As an extension of (1), Malik [7] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$ where $k \le 1$, then

$$n \max_{|z|=1} |P(z)| \le (1 + k) \max_{|z|=1} |P'(z)| . \tag{2}$$

*Mathematics Subject Classifications: 30C10, 30A10, 41A17.

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Equality in (2) holds for $P(z) = (z+k)^n$ where $k \leq 1$.

On the other hand, for the class of polynomials $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, of degree n having all their zeros in $|z| \leq k$, $k \leq 1$, Aziz and Shah [5] proved that

$$n \max_{|z|=1} |P(z)| \leq (1+k^{\mu}) \max_{|z|=1} |P'(z)| - \frac{n}{k^{n-\mu}} \min_{|z|=k} |P(z)|. \tag{3}$$

Malik [8] obtained a generalization of (1) in the sense that the left-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z|=1$. In fact, he proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^q d\theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^q d\theta \right\}^{\frac{1}{q}} \max_{|z|=1} |P'(z)|. \tag{4}$$

The corresponding extension of (2), which is a generalization of (4), was obtained by Aziz [1] who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for each $q \geq 0$

$$n \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^q d\theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^q d\theta \right\}^{\frac{1}{q}} \max_{|z|=1} |P'(z)|. \tag{5}$$

Inequality (5) reduces to the inequality (2) by letting $q \to \infty$, .

As a generalization of (5), Aziz and Ahemad [2] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for each $r > 0$, $p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^{qr} d\theta \right\}^{\frac{1}{qr}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{pr} d\theta \right\}^{\frac{1}{pr}} \tag{6}$$

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n with respect to a point $\alpha \in \mathbb{C}$, then (see [9])

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R$ and $R > 0$.

As an extension of (2) to the polar derivative, Aziz and Rather [3] proved that if all the zeros of $P(z)$ lie in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$n(|\alpha| - k) \max_{|z|=1} |P(z)| \leq (1+k) \max_{|z|=1} |D_\alpha P(z)|. \tag{7}$$

For the class of lacunary type polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, of degree n having all their zeros in $|z| \leq k$ where $k \leq 1$, Aziz and Rather [4] also proved that if for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^\mu$,

$$n(|\alpha| - k^\mu)\max_{|z|=1}|P(z)| \leq (1 + k^\mu)\max_{|z|=1}|D_\alpha P(z)|\,. \tag{8}$$

As a refinement of inequality (8), and an extension of inequality (3) to polar derivative, Rather and Mir [12] proved that if $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^\mu$,

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{n\left(|\alpha| - k^\mu\right)}{1 + k^\mu}\max_{|z|=1}|P(z)| + \frac{n\left(|\alpha| + 1\right)}{k^{n-\mu}\left(1 + k^\mu\right)}\min_{|z|=k}|P(z)|. \tag{9}$$

2 Main Results

In this paper, we first extend inequality (6) to the polar derivative and prove the following result.

THEOREM 1. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and for each $r > 0$, $p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$\begin{aligned} n(|\alpha| - k)&\left\{\int_0^{2\pi}\left|P(e^{i\theta}) + \frac{\beta m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \\ &\leq \left\{\int_0^{2\pi}|1 + ke^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}\left(\left|D_\alpha P(e^{i\theta})\right| - \frac{mn}{k^{n-1}}\right)^{qr}d\theta\right\}^{\frac{1}{qr}} \end{aligned} \tag{10}$$

where $m = \min_{|z|=k}|P(z)|$.

REMARK 1. By letting $r \to \infty$ and choosing the argument of β in the left side of inequality (10) suitably, we obtain a result due to Aziz and Rather [3]. Instead of proving Thereon 1, we prove the following more general result which is also L_r mean extension of (9).

THEOREM 2. If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k^\mu$, $|\beta| \leq 1$ and for each $r > 0$, $p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$\begin{aligned} n(|\alpha| - k^\mu)&\left\{\int_0^{2\pi}\left|P(e^{i\theta}) + \frac{\beta m}{k^{n-\mu}}\right|^r d\theta\right\}^{\frac{1}{r}} \\ &\leq \left\{\int_0^{2\pi}|1 + k^\mu e^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}\left(\left|D_\alpha P(e^{i\theta})\right| - \frac{mn}{k^{n-\mu}}\right)^{qr}d\theta\right\}^{\frac{1}{qr}} \end{aligned} \tag{11}$$

where $m = \min_{|z|=k} |P(z)|$.

If we let $q \to \infty$, in (11) so that $p \to 1$, we obtain the following result.

COROLLARY 1. If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$ where $k \le 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \ge k^\mu$, $|\beta| \le 1$ and for each $r > 0$,

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi} \left|P(e^{i\theta}) + \frac{\beta m}{k^{n-\mu}}\right|^r d\theta\right\}^{\frac{1}{r}}$$

$$\le \left\{\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \left\{\max_{|z|=1} \left|D_\alpha P(z)\right| - \frac{mn}{k^{n-\mu}}\right\} \tag{12}$$

where $m = \min_{|z|=k} |P(z)|$.

REMARK 2. Again, letting $r \to \infty$ and choosing the argument of β in the left side of inequality (12) suitably, we obtain inequality (9).

For the proof of Theorem 2, we need the following Lemma.

3 Lemma

The following Lemma holds due to N. A. Rather [11].

LEMMA 1. If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree almost n having all its zeros in in $|z| \le k$ where $k \le 1$, then for $|z| = 1$,

$$|Q'(z)| + \frac{nm}{k^{n-\mu}} \le k^\mu |P'(z)| \tag{13}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$ and $m = \min_{|z|=k} |P(z)|$.

4 Proof of Theorem 2

In this section, we prove Theorem 2.

Let $Q(z) = z^n \overline{P(1/\overline{z})}$, then $P(z) = z^n \overline{Q(1/\overline{z})}$ and it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \text{ and } |P'(z)| = |nQ(z) - zQ'(z)|. \tag{14}$$

By Lemma 1, we have for every β with $|\beta| \le 1$ and $|z| = 1$,

$$\left|Q'(z) + \bar{\beta}\frac{nmz^{n-1}}{k^{n-\mu}}\right| \le |Q'(z)| + \frac{nm}{k^{n-\mu}} \le k^\mu |P'(z)|. \tag{15}$$

Using (14) in (15), we get for $|z| = 1$,

$$\left|Q'(z) + \bar{\beta}\frac{nmz^{n-1}}{k^{n-\mu}}\right| \le k^\mu |nQ(z) - zQ'(z)|. \tag{16}$$

Again, by Lemma 1 for every real or complex number α with $|\alpha| \geq k$ and $|z| = 1$, we have

$$|D_\alpha P(z)| \geq |\alpha|\,|P'(z)| - |Q'(z)| \geq (|\alpha| - k^\mu)\,|P'(z)| + \frac{mn}{k^{n-\mu}},$$

so that

$$|D_\alpha P(z)| - \frac{mn}{k^{n-\mu}} \geq (|\alpha| - k^\mu)|P'(z)|. \tag{17}$$

Since $P(z)$ has all its zeros in $|z| \leq k \leq 1$, it follows by Gauss-Lucas Theorem that all the zeros of $P'(z)$ also lie in $|z| \leq k \leq 1$. This implies that the polynomial

$$z^{n-1}\overline{P'(1/\overline{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (16) that the function

$$w(z) = \frac{z\left(Q'(z) + \bar{\beta}\dfrac{nmz^{n-1}}{k^{n-\mu}}\right)}{k^\mu\,(nQ(z) - zQ'(z))}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + k^\mu w(z)$ is subordinate to the function $1 + k^\mu z$ for $|z| \leq 1$. Hence by a well known property of subordination [6], we have

$$\int_0^{2\pi} \left|1 + k^\mu w(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^r d\theta,\ r > 0. \tag{18}$$

Now

$$1 + k^\mu w(z) = \frac{n\left(Q(z) + \bar{\beta}\dfrac{mz^n}{k^{n-\mu}}\right)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1}\overline{P'(1/\overline{z})}| = |nQ(z) - zQ'(z)| \text{ for } |z| = 1,$$

therefore for $|z| = 1$,

$$n\left|Q(z) + \bar{\beta}\frac{mz^n}{k^{n-\mu}}\right| = |1 + k^\mu w(z)|\,|nQ(z) - zQ'(z)| = |1 + k^\mu w(z)||P'(z)|.$$

Equivalently,

$$n\left|z^n\overline{P(1/\overline{z})} + \bar{\beta}\frac{mz^n}{k^{n-\mu}}\right| = |1 + k^\mu w(z)||P'(z)|.$$

This implies

$$n\left|P(z) + \beta\frac{m}{k^{n-\mu}}\right| = |1 + k^\mu w(z)||P'(z)| \text{ for } |z| = 1. \tag{19}$$

From (17) and (19), we deduce that for $r > 0$,

$$n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-\mu}}\right|^r d\theta \leq \int_0^{2\pi} \left|1 + k^\mu w(e^{i\theta})\right|^r \left(\left|D_\alpha P(e^{i\theta})\right| - \frac{mn}{k^{n-\mu}}\right)^r d\theta.$$

This gives with the help of Hölder's inequality and (18), for $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-\mu}}\right|^r d\theta$$

$$\leq \left(\int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^{pr} d\theta\right)^{1/p} \left(\int_0^{2\pi} \left\{\left|D_\alpha P(e^{i\theta})\right| - \frac{mn}{k^{n-\mu}}\right\}^{qr} d\theta\right)^{1/q},$$

equivalently,

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-\mu}}\right|^r d\theta\right\}^{\frac{1}{r}}$$

$$\leq \left\{\int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} \left(\left|D_\alpha P(e^{i\theta})\right| - \frac{mn}{k^{n-\mu}}\right)^{qr} d\theta\right\}^{\frac{1}{qr}}$$

which proves the desired result.

Acknowledgment. The authors are highly grateful to the referee for his valuable suggestions and comments. The third author is supported by Council of Scientific and Industrial Research, New Delhi, under grant F.No. 09/251(0047)/2012-EMR-I.

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