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Existence Of Solutions For A Robin Problem Involving The $p(x)$-Laplacian*

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Abstract

We study the existence of weak solutions for a parametric Robin problem driven by the $p(x)$-Laplacian. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces, combined with adequate variational methods and the Mountain Pass Theorem.

1 Introduction

The purpose of this article is to study the existence of solutions for the following problem:

$$\begin{cases} -\Delta_{p(x)}u = \lambda\Big(a(x)|u|^{q(x)-2}u + b(x)|u|^{r(x)-2}u\Big), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, λ is a positive number, p is Lipschitz continuous on $\overline{\Omega}$, $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$, and q, r are continuous functions on $\overline{\Omega}$ with $q^- := \inf_{x\in\overline{\Omega}} q(x) > 1$, $r^- := \inf_{x\in\overline{\Omega}} r(x) > 1$, $a(x), b(x) > 0$ for $x \in \overline{\Omega}$ such that $a \in L^{\alpha(x)}(\Omega)$, $\alpha(x) = \frac{p(x)}{p(x)-q(x)}$ and $b \in L^{\gamma(x)}(\Omega)$, $\gamma(x) = \frac{p^*(x)}{p^*(x)-r(x)}$. Here

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

We will use the notations such as h^- and h^+ where

$$h^- := \inf_{x\in\overline{\Omega}} h(x) \leq h(x) \leq h^+ := \sup_{x\in\overline{\Omega}} h(x) < +\infty.$$

Throughout this paper, assuming the condition

$$1 < q^- \leq q^+ < p^- \leq p^+ < r^- \leq r^+ < (p^-)^* \text{ and } p^+ < N. \tag{2}$$

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The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a natural extension of the classical p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant. However, such generalizations are not trivial since the $p(x)$- Laplace operator possesses a more complicated structure than p Laplace operator; for example, it is inhomogeneous.

Nonlinear boundary value problems with variable exponent have received considerable attention in recent years. This is partly due to their frequent appearance in applications such as the modeling of electro-rheological fluids [12, 13, 17] and image processing [4], but these problems are very interesting from a purely mathematical point of view as well. Many results have been obtained on this kind of problems; see for example [1, 3, 5, 6, 8, 14, 15, 16]. In [5], the authors have studied the case $a(x) = 1$, $b(x) = 0$ and $q(x) = p(x)$, they proved that the existence of infinitely many eigenvalue sequences. Unlike the p-Laplacian case, for a variable exponent $p(x)$ ($\neq$ constant), there does not exist a principal eigenvalue and the set of all eigenvalues is not closed under some assumptions. Finally, they presented some sufficient conditions for the infimum of all eigenvalues to be zero and positive, respectively.

The main result of this paper is as follows.

THEOREM 1. Assume p is Lipschitz continuous, $q, r \in C_+(\overline{\Omega})$ and condition (2) is fulfilled. Then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, problem (1) possesses a nontrivial weak solution.

This article is organized as follows. First, we will introduce some basic preliminary results and lemmas in Section 2. In Section 3, we will give the proof of our main result.

2 Preliminaries

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For more details, see [9, 10]. Suppose that Ω is a bounded open domain of $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$ where

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) \text{ and } \inf_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

Denote by $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbf{R} \text{ is measurable and } \int_\Omega |u|^{p(x)}\, dx < +\infty \right\},$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \tau > 0; \int_\Omega \left|\frac{u}{\tau}\right|^{p(x)} dx \leq 1 \right\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \},$$

with the norm

$$\|u\| = \inf\left\{\tau > 0 : \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \left|\frac{u}{\tau}\right|^{p(x)}\right) dx \le 1\right\},$$

$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}\,.$$

We refer the reader to [8, 9] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

LEMMA 1 (cf. [10]). Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable and uniformly convex Banach spaces.

LEMMA 2 (cf. [10]). Hölder inequality holds, namely

$$\int_\Omega |uv|\,dx \le 2\,|u|_{p(x)}\,|v|_{p'(x)} \quad \text{for all } u \in L^{p(x)}(\Omega) \text{ and } v \in L^{p'(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

LEMMA 3 (cf. [2]). Assume that $h \in L^\infty_+(\Omega)$ and $p \in C_+(\overline{\Omega})$. If $|u|^{h(x)} \in L^{p(x)}(\Omega)$, then we have

$$\min\left\{|u|^{h^-}_{h(x)p(x)}, |u|^{h^+}_{h(x)p(x)}\right\} \le \left||u|^{h(x)}\right|_{p(x)} \le \max\left\{|u|^{h^-}_{h(x)p(x)}, |u|^{h^+}_{h(x)p(x)}\right\}.$$

LEMMA 4 (cf. [9]). Assume that Ω is bounded and smooth.

(i) If p is Lipschitz continuous and $p^+ < N$, then for $h \in L^\infty_+(\Omega)$ with $p(x) \le h(x) \le p^*(x)$ there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega)$.

(ii) If $p \in C(\overline{\Omega})$ and $1 \le q(x) < p^*(x)$ for $x \in \overline{\Omega}$ where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \ge N, \end{cases}$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Now, we introduce a norm, which will be used later. Let $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$ and for $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_\beta = \inf\left\{\tau > 0 : \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} dx + \int_{\partial\Omega} \beta(x)\left|\frac{u}{\tau}\right|^{p(x)}\right) d\sigma \le 1\right\}.$$

Then, by Theorem 2.1 in [7], $\|.\|_\beta$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|.\|$.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping defined by the following.

LEMMA 5 (cf. [7]). Let $I_\beta(u) = \int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)\,|u|^{p(x)}\,d\sigma$ with $\beta^- > 0$. For $u \in W^{1,p(x)}(\Omega)$ we have that

(i) $\|u\|_\beta < 1(=1, >1) \Leftrightarrow I_\beta(u) < 1(=1, >1)$,

(ii) $\|u\|_\beta \le 1 \Rightarrow \|u\|_\beta^{p^+} \le I_\beta(u) \le \|u\|_\beta^{p^-}$, and

(iii) $\|u\|_\beta \ge 1 \Rightarrow \|u\|_\beta^{p^-} \le I_\beta(u) \le \|u\|_\beta^{p^+}$.

Here, problem (1) is stated in the framework of the generalized Sobolev space $X := W^{1,p(x)}(\Omega)$.

The Euler-Lagrange functional associated with (1) is defined as $\Phi_\lambda : X \to \mathbb{R}$,

$$\begin{aligned}\Phi_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)}\, d\sigma - \lambda \int_\Omega \frac{a(x)}{q(x)} |u|^{q(x)}\, dx \\ &\quad -\lambda \int_\Omega \frac{b(x)}{r(x)} |u|^{r(x)}\, dx.\end{aligned}$$

We say that $u \in X$ is a weak solution of (1) if

$$\begin{aligned}&\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx + \int_{\partial\Omega} \beta(x) |u|^{p(x)-2} uv d\sigma \\ &= \lambda \int_\Omega a(x) |u|^{q(x)-2} uv dx + \lambda \int_\Omega b(x) |u|^{r(x)-2} uv dx\end{aligned}$$

for all $v \in X$.

Standard arguments imply that $\Phi_\lambda \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}\langle \Phi_\lambda'(u), v\rangle &= \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx + \int_{\partial\Omega} \beta(x) |u|^{p(x)-2} uv d\sigma \\ &\quad - \lambda \int_\Omega a(x) |u|^{q(x)-2} uv dx - \lambda \int_\Omega b(x) |u|^{r(x)-2} uv dx,\end{aligned}$$

for all $u, v \in X$. Thus the weak solutions of (1) coincide with the critical points of Φ_λ. If such a weak solution exists and is nontrivial, then the corresponding λ is an eigenvalue of problem (1).

Next, we write Φ_λ' as

$$\Phi_\lambda' = A - \lambda B,$$

where $A, B : X \to X'$ are defined by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx + \int_{\partial\Omega} \beta(x) |u|^{p(x)-2} uv\, d\sigma$$

and

$$\langle B(u), v\rangle = \int_\Omega a(x) |u|^{q(x)-2} uv\, dx + \int_\Omega b(x) |u|^{r(x)-2} uv\, dx.$$

Denote by $M, C, C_i, i = 1, 2...$ the general positive constants which are the exact values may change from line to line.

LEMMA 6 (cf. [11]). A satisfies condition (S^+), namely, $u_n \rightharpoonup u$, in X and $\limsup \langle A(u_n), u_n - u\rangle \le 0$, imply $u_n \to u$ in X.

REMARK 1. Noting that Φ_λ' is still of type (S^+). Hence, any bounded (PS) sequence of Φ_λ in the reflexive Banach space X has a convergent subsequence.

3 Proof of Main Result

For the proof of our theorem, we will use the Mountain Pass Lemma. We need to establish some lemmas.

LEMMA 7. The functional Φ_λ satisfies the Palais-Smale condition (PS).

PROOF. Suppose that $(u_n) \subset X$ is a (PS) sequence; that is,

$$\sup |\phi_\lambda(u_n)| \leq M \text{ (for any } n \text{ or as } n \to \infty \text{ ?) and } \phi'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty.$$

Let us show that (u_n) is bounded in X. Assume $\|u_n\|_\beta > 1$ for convenience. Since $\phi_\lambda(u_n)$ is bounded, we have for n large enough:

$$\begin{aligned}
M+1 &\geq \phi_\lambda(u_n) - \frac{1}{r^-}\langle \phi'_\lambda(u_n), u_n\rangle + \frac{1}{r^-}\langle \phi'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}\,|\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}\,|u_n|^{p(x)}\ d\sigma - \lambda \int_\Omega \frac{a(x)}{q(x)}|u_n|^{q(x)}dx \\
&\quad -\lambda \int_\Omega \frac{b(x)}{r(x)}|u_n|^{r(x)}dx - \frac{1}{r^-}\Big(\int_\Omega |\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)}\,d\sigma\Big) \\
&\quad + \frac{\lambda}{r^-}\int_\Omega a(x)|u_n|^{q(x)}dx + \frac{\lambda}{r^-}\int_\Omega b(x)\,|u_n|^{r(x)}\ dx + \frac{1}{r^-}\langle \phi'_\lambda(u_n), u_n\rangle \\
&\geq \frac{1}{p^+}I_\beta(u_n) - \frac{\lambda}{q^-}\int_\Omega a(x)|u_n|^{q(x)}dx - \frac{\lambda}{r^-}\int_\Omega b(x)|u_n|^{r(x)}dx - \frac{1}{r^-}I_\beta(u_n) \\
&\quad + \frac{\lambda}{r^-}\int_\Omega a(x)|u_n|^{q(x)}dx + \frac{\lambda}{r^-}\int_\Omega b(x)|u_n|^{r(x)}dx + \frac{1}{r^-}\langle \phi'_\lambda(u_n), u_n\rangle \\
&\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) I_\beta(u_n) - \lambda\left(\frac{1}{q^-} - \frac{1}{r^-}\right) C_1 |a|_{\alpha(x)}\,\|u_n\|^{q^+} \\
&\quad - \frac{1}{r^-}\,\left\|\phi'_\lambda(u_n)\right\|_{X'}\,\|u_n\| \\
&\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\|u_n\|_\beta^{p^-} - \lambda\left(\frac{1}{q^-} - \frac{1}{r^-}\right) C_1\,|a|_{\alpha(x)}\,\|u_n\|^{q^+} - \frac{C_2}{r^-}\|u_n\| \\
&\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) C_3\,\|u_n\|^{p^-} - \lambda\left(\frac{1}{q^-} - \frac{1}{r^-}\right) C_1|a|_{\alpha(x)}\|u_n\|^{q^+} - \frac{C_2}{r^-}\|u_n\|,
\end{aligned}$$

hence (u_n) is bounded in X since $q^- \leq q^+ < p^- \leq p^+ < r^-$. The proof is completed.

LEMMA 8. There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\rho, \tau > 0$ such that $\Phi_\lambda(u) \geq \tau > 0$ for any $u \in X$ with $\|u\|_\beta = \rho$.

PROOF. Using Lemma 4, there exists a positive constant C_4 such that

$$|u|_{p(x)} \leq C_4\,\|u\|_\beta \text{ and } |u|_{p^*(x)} \leq C_4\,\|u\|_\beta \text{ for all } u \in X. \tag{3}$$

Fix $\rho \in]0,1[$ such that $\rho < \frac{1}{C_4}$. Then relation (3) implies $|u|_{p(x)} < 1$, $|u|_{p^*(x)} < 1$, for all $u \in X$ with $\|u\|_\beta = \rho$. Using Lemmas 2 and 3, we obtain

$$\int_\Omega a(x)\,|u|^{q(x)}\,dx \le 2|a|_{\alpha(x)} \left| |u|^{q(x)} \right|_{\frac{p(x)}{q(x)}} \le 2\,|a|_{\alpha(x)}\,|u|^{q^-}_{p(x)}\,, \tag{4}$$

and

$$\int_\Omega b(x)|u|^{r(x)}dx \le 2\,|b|_{\gamma(x)}\,| \left| u|^{r(x)} \right|_{\frac{p^*(x)}{r(x)}} \le 2|b|_{\gamma(x)}\,|u|^{r^-}_{p^*(x)}\,, \tag{5}$$

for all $u \in X$ with $\|u\|_\beta = \rho$. Combining (3), (4) and (5), we obtain

$$\int_\Omega a(x)|u|^{q(x)}dx \le 2|a|_{\alpha(x)} C_4^{q^-}\,\|u\|_\beta^{q^-}\,, \tag{6}$$

and

$$\int_\Omega b(x)|u|^{r(x)}dx \le 2|b|_{\gamma(x)} C_4^{r^-}\,\|u\|_\beta^{r^-}\,, \tag{7}$$

for all $u \in X$ with $\|u\|_\beta = \rho$. Hence, from (6), (7) we deduce that for any $u \in X$ with $\|u\|_\beta = \rho$, we have

$$\begin{aligned}
\Phi_\lambda(u) \quad &\ge \quad \frac{1}{p^+}\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}d\sigma\right) - \frac{\lambda}{q^-}\int_\Omega a(x)\,|u|^{q(x)}\,dx \\
&\quad - \frac{\lambda}{r^-}\int_\Omega b(x)\,|u|^{r(x)}\,dx \\
&\ge \quad \frac{1}{p^+}\|u\|_\beta^{p^+} - \frac{\lambda}{q^-}2|a|_{\alpha(x)}C_4^{q^-}\,\|u\|_\beta^{q^-} - \frac{\lambda}{r^-}2|b|_{\gamma(x)}C_4^{r^-}\,\|u\|_\beta^{r^-}\,.
\end{aligned}$$

Putting

$$\lambda^* = \min\left\{ \frac{q^-\rho^{p^+-q^-}}{8C_4^{q^-}p^+|a|_{\alpha(x)}}, \frac{r^-\rho^{p^+-r^-}}{8C_4^{r^-}p^+|b|_{\gamma(x)}} \right\} \tag{8}$$

for any $u \in X$ with $\|u\|_\beta = \rho$, there exists $\tau = \rho^{p^+}/(2p^+)$ such that

$$\Phi_\lambda(u) \ge \tau > 0.$$

This completes the proof.

LEMMA 9. There exists $\xi \in X$ such that $\xi \ge 0$, $\xi \ne 0$ and $\Phi_\lambda(t\xi) < 0$, for $t > 0$ small enough.

PROOF. Let $\xi \in C_0^\infty(\Omega)$, $\xi \geq 0$, $\xi \neq 0$ and $t \in (0,1)$. We have

$$\begin{aligned}
\Phi_\lambda(t\xi) &= \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \xi|^{p(x)} dx + \int_{\partial\Omega} \frac{t^{p(x)}\beta(x)}{p(x)} |\xi|^{p(x)} \, d\sigma - \lambda \int_\Omega a(x) \frac{t^{q(x)}}{q(x)} |\xi|^{q(x)} dx \\
&\quad -\lambda \int_\Omega b(x) \frac{t^{r(x)}}{r(x)} |\xi|^{r(x)} dx \\
&\leq \frac{t^{p^-}}{p^-} \left(\int_\Omega |\nabla \xi|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |\xi|^{p(x)} \, d\sigma \right) - \frac{\lambda t^{q^+}}{q^+} \int_\Omega a(x) |\xi|^{q(x)} dx \\
&\quad - \frac{\lambda t^{r^+}}{r^+} \int_\Omega b(x) |\xi|^{r(x)} dx \\
&\leq \frac{t^{p^-}}{p^-} \left(\int_\Omega |\nabla \xi|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |\xi|^{p(x)} \, d\sigma \right) \\
&\quad - \frac{\lambda t^{q^+}}{q^+} \left(\int_\Omega a(x) |\xi|^{q(x)} dx + \int_\Omega b(x) |\xi|^{r(x)} dx \right).
\end{aligned}$$

Then, for any $t < \delta^{\frac{1}{p^- - q^+}}$, with

$$0 < \delta < \min \left\{ 1, \frac{\lambda p^- \left(\int_\Omega a(x) |\xi|^{q(x)} dx + \int_\Omega b(x) |\xi|^{r(x)} dx \right)}{q^+ \left(\int_\Omega |\nabla \xi|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |\xi|^{p(x)} \, d\sigma \right)} \right\},$$

we conclude that

$$\Phi_\lambda(t\xi) < 0.$$

The proof is complete.

We now turn to the proof of Theorem 1. To apply the Mountain Pass Theorem, we need to prove that

$$\phi(tu) \to -\infty \text{ as } t \to +\infty,$$

for a certain $u \in X$. Let $\omega \in C_0^\infty(\Omega)$, $\omega \geq 0$, $\omega \neq 0$ and $t > 1$. We have

$$\begin{aligned}
\Phi_\lambda(t\omega) &= \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \omega|^{p(x)} dx + \int_{\partial\Omega} \frac{t^{p(x)}\beta(x)}{p(x)} |\omega|^{p(x)} \, d\sigma - \lambda \int_\Omega a(x) \frac{t^{q(x)}}{q(x)} |\omega|^{q(x)} dx \\
&\quad -\lambda \int_\Omega b(x) \frac{t^{r(x)}}{r(x)} |\omega|^{r(x)} dx \\
&\leq \frac{t^{p^+}}{p^-} \left(\int_\Omega |\nabla \omega|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |\omega|^{p(x)} \, d\sigma \right) - \frac{\lambda t^{q^-}}{q^+} \int_\Omega a(x) |\omega|^{q(x)} dx \\
&\quad - \frac{\lambda t^{r^-}}{r^+} \int_\Omega b(x) |\omega|^{r(x)} dx.
\end{aligned}$$

Since $q^-, p^+ < r^-$ we have $\phi(t\omega) \to -\infty$ as $t \to +\infty$. It follows that there exists $e \in X$ such that $\|e\|_\beta > \rho$ and $\phi_\lambda(e) < 0$. According to the Mountain Pass Theorem, ϕ_λ admits a critical value $\theta \geq \tau$ which is characterized by

$$\theta = \inf_{g \in \Gamma} \sup_{t \in [0,1]} \phi_\lambda(g(t)),$$

where

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0 \text{ and } g(1) = e\}.$$

This completes the proof.

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