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Computing The Determinants By Reducing The Orders By Four*

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Abstract

We present a new method to compute the determinants of $n \times n\,(n \geq 5)$ matrices by reducing their sizes by four. To prove our results we use the so-called "cornice determinants", i.e, square determinants of order $n\,(n \geq 5)$ where, with the exception of the first and last entries, the entries of the 2nd row and $(n-1)$-th row, as well the 2nd column and $(n-1)$-th column are all zero. The method introduced here has the advantage of reducing the size of a determinant by four, and thus enabling their quicker and easier computation.

1 Introduction

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

DEFINITION 1. A determinant of order n, or size $n \times n$, (see [5], [9], [10], [11]) is the sum

$$D = \det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{S_n} \varepsilon_{j_1,j_2,\ldots,j_n} a_{j_1} a_{j_2} \ldots a_{j_n},$$

ranging over the symmetric permutation group S_n, where

$$\varepsilon_{j_1,j_2,\ldots,j_n} = \begin{cases} +1, \text{ if } j_1, j_2, \ldots, j_n \text{ is an even permutation} \\ -1, \text{ if } j_1, j_2, \ldots, j_n \text{ is an odd permuation.} \end{cases}$$

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1.1 Properties Characterizing Determinants

Let A and B be any $n \times n$ matrices.

1. If A is a triangular matrix, i.e. $a_{ij} = 0$ whenever $i > j$ or, alternatively, whenever $i < j$, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.
2. If B results from A by interchanging two rows or columns, then $\det(B) = -\det(A)$.
3. If B results from A by multiplying one row or column with a number c, then $\det(B) = c\det(A)$.
4. If B results from A by adding a multiple of one row to another row, or a multiple of one column to another column, then $\det(B) = \det(A)$.

These four properties (see e.g. [3], [7], [8]) can be used to compute the determinant of any matrix, using Gaussian elimination. This is an algorithm that transforms any given matrix to a triangular matrix, only by using the operations from the last three items above. Since the effect of these operations on the determinant can be traced, the determinant of the original matrix is known, once Gaussian elimination is performed.

It is also possible to expand a determinant along a row or column using Laplace's formula, which is efficient for relatively small matrices. To do this along the row i, say, we write

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij},$$

where the C_{ij} represent the matrix cofactors, i.e., C_{ij} is $(-1)^{i+j}$ times the minor M_{ij}, which is the determinant of the matrix that results from A by removing the i-th row and the j-th column, and n is the size of the matrix.

5. If I is identity matrix, i.e.

   $$a_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases},$$

   then the determinant of the identity matrix is one, $\det(I) = 1$.
6. Multiplying a matrix by a number r affects the determinant as follows: $\det(rA) = r^n \det(A)$.
7. If A contains a zero row (or column), then $\det(A) = 0$.
8. If A contains two identical or proportional rows (or columns), then $\det(A) = 0$.
9. A matrix and its transpose have the same determinant: $\det(A^T) = \det(A)$.

PROPOSITION 1. Let B be obtained from A by one of the following elementary row (column) operations:

1) Two rows (or columns) of A are switched, or

2) A row (or column) of A is multiplied by a number α, or

3) A multiplier of a row (or column) of A is added to another row (or column).

Then we have, respectively, $\det(B) = -\det(A)$, or $\det(B) = \alpha \det(A)$, or $\det(B) = \det(A)$.

1.2 Chio's Condensation Method

Chio's condensation is a method for evaluating an $n \times n$ determinant in terms of $(n-1) \times (n-1)$ determinants; see [1], [4]:

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \frac{1}{a_{11}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}.$$

1.3 Dodgson Condensation Method

Dodgson's condensation method computes determinants of size $n \times n$ by expressing them in terms of those of size $(n-1) \times (n-1)$, and then expresses the latter in terms of determinants of size $(n-2) \times (n-2)$, and so on (see [2]).

2 A New Method

Let $|A_{n\times n}|$ be an $n \times n$ determinant:

$$|A_{n\times n}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{1}$$

Let us now analyze determinants of size $n \times n \, (n \geq 5)$, where, with the exception of the first and last entries, the entries of the 2nd row and $(n-1)$-th row, as well the 2nd column and $(n-1)$-th column are all zero. We call such determinants "cornice determinants" and write them down as $|C_{n\times n}|$ $(n \geq 5)$:

$$|C_{n\times n}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\ a_{21} & 0 & 0 & \cdots & 0 & 0 & a_{2n} \\ a_{31} & 0 & a_{33} & \cdots & a_{3,n-2} & 0 & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,1} & 0 & a_{n-2,3} & \cdots & a_{n-2,n-2} & 0 & a_{n-2,n} \\ a_{n-1,1} & 0 & 0 & \cdots & 0 & 0 & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{vmatrix} \qquad (2)$$

By using elementary row and column operations and based on determinants properties, a considerable number of determinants of size $n \times n$ can be transformed into cornice determinants.

We will prove that any cornice determinant of size n $(n \geq 5)$ can be computed by reducing its size by four, thus transforming it to an $(n-4)\times(n-4)$ determinant. For example, a determinant of size 5×5 is transformed to a number, one of size 6×6 to a determinant of size 2×2, etc.

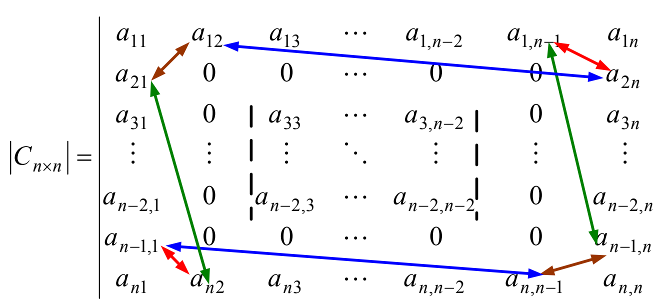
THEOREM 1. Every cornice determinant $|C_{n\times n}|$ of size $n \times n$ $(n \geq 5)$ can be computed by reducing the order of the determinant by four:

$$\begin{aligned} |C_{n\times n}| &= (a_{12}a_{21}a_{n,n-1}a_{n-1,n} - a_{12}a_{2n}a_{n,n-1}a_{n-1,1} \\ &\quad -a_{21}a_{n2}a_{1,n-1}a_{n-1,n} + a_{1,n-1}a_{2n}a_{n2}a_{n-1,1})|C_{(n-4)\times(n-4)}|, \end{aligned} \qquad (3)$$

where

$$|C_{(n-4)\times(n-4)}| = \begin{vmatrix} a_{33} & \cdots & a_{3,n-2} \\ \vdots & \ddots & \vdots \\ a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix}.$$

We illustrate this using the following picture:



PROOF. Let $n = 5$. We will prove that Theorem 1 holds for $|C_{5\times 5}|$ cornice determinants. Based on Laplace's formula, the determinant $|C_{5\times 5}|$ can be expanded along the first row: $|C_{5\times 5}|$ is equal to

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & 0 & 0 & 0 & a_{25} \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ a_{41} & 0 & 0 & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}$$

$$\begin{aligned} = \quad & a_{11}\begin{vmatrix} 0 & 0 & 0 & a_{25} \\ 0 & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & 0 & 0 & a_{25} \\ a_{31} & a_{33} & 0 & a_{35} \\ a_{41} & 0 & 0 & a_{45} \\ a_{51} & a_{53} & a_{54} & a_{55} \end{vmatrix} \\ & +a_{13}\begin{vmatrix} a_{21} & 0 & 0 & a_{25} \\ a_{31} & 0 & 0 & a_{35} \\ a_{41} & 0 & 0 & a_{45} \\ a_{51} & a_{52} & a_{54} & a_{55} \end{vmatrix} + +(-1)^{1+4}a_{14}\begin{vmatrix} a_{21} & 0 & 0 & a_{25} \\ a_{31} & 0 & a_{33} & a_{35} \\ a_{41} & 0 & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{55} \end{vmatrix} \\ & +a_{15}\begin{vmatrix} a_{21} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix}. \end{aligned}$$

Based on property 8 from the first section, the first, third and fifth determinants are zero, and hence

$$\begin{aligned} |C_{5\times 5}| \quad = \quad & -a_{12}\left((-1)^{1+1}a_{21}\begin{vmatrix} a_{33} & 0 & a_{35} \\ 0 & 0 & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix} + (-1)^{1+4}a_{25}\begin{vmatrix} a_{31} & a_{33} & 0 \\ a_{41} & 0 & 0 \\ a_{51} & a_{52} & a_{54} \end{vmatrix}\right) \\ & -a_{14}\left((-1)^{1+1}a_{21}\begin{vmatrix} 0 & a_{33} & a_{35} \\ 0 & 0 & a_{45} \\ a_{52} & a_{53} & a_{55} \end{vmatrix} + (-1)^{1+4}a_{25}\begin{vmatrix} a_{31} & 0 & a_{33} \\ a_{41} & 0 & 0 \\ a_{51} & a_{52} & a_{53} \end{vmatrix}\right) \\ = \quad & -a_{12}\left[a_{21}(-a_{33}a_{45}a_{54}) - a_{25}(-a_{33}a_{41}a_{54})\right] - a_{14}\left[a_{21}(a_{33}a_{45}a_{52}) - a_{25}(a_{33}a_{41}a_{52})\right] \\ = \quad & a_{12}a_{21}a_{33}a_{45}a_{54} - a_{12}a_{25}a_{33}a_{41}a_{54} - a_{14}a_{21}a_{33}a_{45}a_{52} + a_{14}a_{25}a_{33}a_{41}a_{52} \\ = \quad & (a_{12}a_{21}a_{54}a_{45} - a_{12}a_{25}a_{54}a_{41} - a_{21}a_{52}a_{14}a_{45} + a_{41}a_{52}a_{25}a_{14})\, a_{33}, \end{aligned}$$

which has the desired form.

Next, we prove that the theorem holds for $n \geq 6$. Based on Laplace's formula, the determinant $|C_{n\times n}|$ can be expanded along the second column to yield:

$$
\begin{aligned}
&\begin{vmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\
a_{21} & 0 & 0 & \cdots & 0 & 0 & a_{2n} \\
a_{31} & 0 & a_{33} & \cdots & a_{3,n-2} & 0 & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{n-2,1} & 0 & a_{n-2,3} & \cdots & a_{n-2,n-2} & 0 & a_{n-2,n} \\
a_{n-1,1} & 0 & 0 & \cdots & 0 & 0 & a_{n-1,n} \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-2} & a_{n,n-1} & a_{nn}
\end{vmatrix} \\
= \quad & (-1)^{1+2}a_{12}\begin{vmatrix}
a_{21} & 0 & \cdots & 0 & 0 & a_{2n} \\
a_{31} & a_{33} & \cdots & a_{3,n-2} & 0 & a_{3n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{n-2,1} & a_{n-2,3} & \cdots & a_{n-2,n-2} & 0 & a_{n-2,n} \\
a_{n-1,1} & 0 & \cdots & 0 & 0 & a_{n-1,n} \\
a_{n1} & a_{n3} & \cdots & a_{n,n-2} & a_{n,n-1} & a_{nn}
\end{vmatrix} \\
& +(-1)^{n+2}a_{n2}\begin{vmatrix}
a_{11} & a_{13} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\
a_{21} & 0 & \cdots & 0 & 0 & a_{2n} \\
a_{31} & a_{33} & \cdots & a_{3,n-2} & 0 & a_{3n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{n-2,1} & a_{n-2,3} & \cdots & a_{n-2,n-2} & 0 & a_{n-2,n} \\
a_{n-1,1} & 0 & \cdots & 0 & 0 & a_{n-1,n}
\end{vmatrix}. \qquad (4)
\end{aligned}
$$

The determinants in (4) are expanded along the $(n-1)$-th column so that $|C_{n\times n}|$ is equal to

$$
\begin{aligned}
& (-1)^{1+2}a_{12}(-1)^{n-1+n-2}a_{n,n-1}\begin{vmatrix}
a_{21} & 0 & \cdots & 0 & a_{2n} \\
a_{31} & a_{33} & \cdots & a_{3,n-2} & a_{3n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-2,1} & a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n} \\
a_{n-1,1} & 0 & \cdots & 0 & a_{n-1,n}
\end{vmatrix} \\
& +(-1)^{n+2}a_{n2}(-1)^{1+n-2}a_{1,n-1}\begin{vmatrix}
a_{21} & 0 & \cdots & 0 & a_{2n} \\
a_{31} & a_{33} & \cdots & a_{3,n-2} & a_{3n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-2,1} & a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n} \\
a_{n-1,1} & 0 & \cdots & 0 & a_{n-1,n}
\end{vmatrix} \\
= \quad & \big((-1)^{2n}a_{12}a_{n,n-1} + (-1)^{2n+1}a_{n2}a_{1,n-1}\big)\begin{vmatrix}
a_{21} & 0 & \cdots & 0 & a_{2n} \\
a_{31} & a_{33} & \cdots & a_{3,n-2} & a_{3n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-2,1} & a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n} \\
a_{n-1,1} & 0 & \cdots & 0 & a_{n-1,n}
\end{vmatrix}.
\end{aligned}
$$

The determinant in the right hand side is expanded along the first row to yield

$$(a_{12}a_{n,n-1}-a_{n2}a_{1,n-1})\left[(-1)^{1+1}a_{21}\begin{vmatrix} a_{33} & \cdots & a_{3,n-2} & a_{3n}\\ \vdots & \ddots & \vdots & \vdots\\ a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n}\\ 0 & \cdots & 0 & a_{n-1,n}\end{vmatrix}+\right.$$

$$\left.+(-1)^{1+n-2}a_{2n}\begin{vmatrix} a_{31} & a_{33} & \cdots & a_{3,n-2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n-2,1} & a_{n-2,3} & \cdots & a_{n-2,n-2}\\ a_{n-1,1} & 0 & \cdots & 0\end{vmatrix}\right]. \tag{5}$$

The determinants in (5) are expanded along the last row to yield

$$\begin{aligned}
&(a_{12}a_{n,n-1}-a_{n2}a_{1,n-1})\left(a_{21}(-1)^{n-3+n-3}a_{n-1,n}+(-1)^{n-1}a_{2n}(-1)^{n-2}a_{n-1,1}\right)\\
&\times\begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3,n-2}\\ a_{43} & a_{44} & \cdots & a_{4,n-2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n-2,3} & a_{n-2,4} & \cdots & a_{n-2,n-2}\end{vmatrix}\\
=\;&(a_{12}a_{n,n-1}-a_{n2}a_{1,n-1})\,(a_{21}a_{n-1,n}-a_{2n}a_{n-1,1})\\
&\times\begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3,n-2}\\ a_{43} & a_{44} & \cdots & a_{4,n-2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n-2,3} & a_{n-2,4} & \cdots & a_{n-2,n-2}\end{vmatrix}\\
=\;&(a_{12}a_{21}a_{n,n-1}a_{n-1,n}-a_{12}a_{2n}a_{n,n-1}a_{n-1,1}\\
&-a_{21}a_{n2}a_{1,n-1}a_{n-1,n}+a_{1,n-1}a_{2n}a_{n2}a_{n-1,1})|C_{(n-4)\times(n-4)}|,
\end{aligned}$$

which proves that Theorem 1 holds.

We remark that if a matrix A of size $n\times n$ ($n\geq 5$) can be transformed into a cornice determinant $|C_{n\times n}|$, then it can be computed using the formula (3).

To illustrate the usefulness of our method in computing cornice determinants, we give an example:

$$\begin{aligned}
|C_{6\times 6}| &= \begin{vmatrix} 7 & 1 & -3 & 11 & 9 & -6\\ 8 & 0 & 0 & 0 & 0 & 2\\ -5 & 0 & 5 & -5 & 0 & 1\\ 12 & 0 & 3 & 7 & 0 & 2\\ 3 & 0 & 0 & 0 & 0 & 11\\ 10 & 9 & -2 & 1 & 5 & 7\end{vmatrix}\\
&= [8\cdot 1\cdot 5\cdot 11-1\cdot 2\cdot 3\cdot 5-8\cdot 9\cdot 9\cdot 11+9\cdot 2\cdot 3\cdot 9]\begin{vmatrix} 5 & -5\\ 3 & 7\end{vmatrix}\\
&= (440-30-7128+486)\cdot 50=-311600.
\end{aligned}$$

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