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A Tail Bound For Sums Of Independent Random Variables And Application To The Pareto Distribution*

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Abstract

We prove a bound of the tail probability for a sum of n independent random variables. It can be applied under mild assumptions; the variables are not assumed to be almost surely absolutely bounded, or admit finite moments of all orders. In some cases, it is better than the bound obtained via the Fuk-Nagaev inequality. To illustrate this result, we investigate the bound of the tail probability for a sum of n weighted i.i.d. random variables having the symmetric Pareto distribution.

1 Introduction

Let $(Y_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables. For any $n \in \mathbb{N}^*$, we wish to determine the smallest sequence of functions $p_n(t)$ such that $\mathbb{P}\left(\sum_{i=1}^{n} Y_i \geq t\right) \leq p_n(t)$, $t \in [0, \infty[$. This problem is well-known; numerous results exist. The most famous of them is the Markov inequality. Under mild assumptions on the moments of the X_i's, it gives a polynomial bound $p_n(t)$. Under the same assumptions, this bound can be improved by the Fuk-Nagaev inequality (see [2]). If the X_i's are almost surely absolutely bounded, or admit finite moments of all orders (and these moments satisfy some inequalities), the Bernstein inequalities provide better results. See [6] and [7] for further details.

In this note, we present a new bound $p_n(t)$. It can be applied under mild assumptions on the X_i's; only knowledge of the order of a finite moment is required. The main interest of the proposed inequality is that it can be applied when the 'Bernstein conditions' are not satisfied, and can give better results than the Fuk-Nagaev inequality (and the Markov inequality). The tail probability for a sum of n weighted i.i.d. random variables having the symmetric Pareto distribution is studied. This is particularly interesting because the exact expression of the distribution of such a sum is really difficult to identify (see [8]). Moreover, there are some applications in economics, actuarial science, survival analysis and queuing networks.

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The note is organized as follows. Section 2 presents a general tail bound. An application of this bound to the Pareto distribution can be found in Section 3. Section 4 provides a comparative study of the considered inequality and the Fuk-Nagaev inequality.

2 A General Tail Bound

Theorem 1 below presents a bound of the tail probability for a sum of n independent random variables. As for the Fuk-Nagaev inequality (and the Markov inequality), it requires knowledge only of the order of a finite moment.

THEOREM 1. Let $(Y_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables. We suppose that

- for any $n \in \mathbb{N}^*$ and any $i \in \{1,\dots,n\}$, $\mathbb{E}(Y_i) = 0$,
- there exists a real number $p \geq 2$ such that, for any $n \in \mathbb{N}^*$ and any $i \in \{1,\dots,n\}$, $\mathbb{E}(|Y_i|^p) < \infty$.

Then, for any $t > 0$ and any $n \in \mathbb{N}^*$, we have

$$\mathbb{P}\left(\sum_{i=1}^{n} Y_i \geq t\right) \leq C_p t^{-p} \max\left(r_{n,p}(t), (r_{n,2}(t))^{p/2}\right) + \exp\left(-\frac{t^2}{16 b_n}\right), \qquad (1)$$

where, for any $u \in \{2,p\}$, $r_{n,u}(t) = \sum_{i=1}^{n} \mathbb{E}\left(|Y_i|^u 1_{\left\{|Y_i| \geq \frac{3b_n}{t}\right\}}\right)$, $b_n = \sum_{i=1}^{n} \mathbb{E}\left(Y_i^2\right)$, $C_p = 2^{2p} c_p$, and c_p refers to the Rosenthal inequality (see Lemma 1 below).

PROOF. Let $n \in \mathbb{N}^*$. For any $t > 0$, we have

$$\mathbb{P}\left(\sum_{i=1}^{n} Y_i \geq t\right) = \mathbb{P}\left(\sum_{i=1}^{n} (Y_i - \mathbb{E}(Y_i)) \geq t\right) \leq U + V,$$

where

$$U = \mathbb{P}\left(\sum_{i=1}^{n}\left(Y_i 1_{\left\{|Y_i| \geq \frac{3b_n}{t}\right\}} - \mathbb{E}\left(Y_i 1_{\left\{|Y_i| \geq \frac{3b_n}{t}\right\}}\right)\right) \geq \frac{t}{2}\right)$$

and

$$V = \mathbb{P}\left(\sum_{i=1}^{n}\left(Y_i 1_{\left\{|Y_i| < \frac{3b_n}{t}\right\}} - \mathbb{E}\left(Y_i 1_{\left\{|Y_i| < \frac{3b_n}{t}\right\}}\right)\right) \geq \frac{t}{2}\right).$$

Let us bound U and V, in turn.

The upper bound for U. The Markov inequality yields

$$U \leq 2^p t^{-p} \mathbb{E}\left(\left|\sum_{i=1}^{n}\left(Y_i 1_{\left\{|Y_i| \geq \frac{3b_n}{t}\right\}} - \mathbb{E}\left(Y_i 1_{\left\{|Y_i| \geq \frac{3b_n}{t}\right\}}\right)\right)\right|^p\right). \qquad (2)$$

Now, let us introduce the Rosenthal inequality (see [9]).

LEMMA 1 (Rosenthal's inequality). Let $p \geq 2$ and $(X_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables such that, for any $n \in \mathbb{N}^*$ and any $i \in \{1,\dots,n\}$, $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(|X_i|^p) < \infty$. Then we have

$$\mathbb{E}\left(\left|\sum_{i=1}^{n} X_i\right|^p\right) \leq c_p \max\left(\sum_{i=1}^{n}\mathbb{E}\left(|X_i|^p\right), \left(\sum_{i=1}^{n}\mathbb{E}\left(X_i^2\right)\right)^{p/2}\right),$$

where, for any $\tau > p/2$, $c_p = 2\max\left(\tau^p, p\tau^{p/2}e^\tau \int_0^\infty x^{p/2-1}(1-x)^{-\tau}dx\right)$.

Under mild assumptions on the X_i's, the constant c_p of Lemma 1 can be improved. We refer to [1], [4] and [10].

For any $i \in \{1,\dots,n\}$, set $Z_i = Y_i 1_{\{|Y_i|\geq\frac{3b_n}{t}\}} - \mathbb{E}\left(Y_i 1_{\{|Y_i|\geq\frac{3b_n}{t}\}}\right)$. Since $\mathbb{E}(Z_i) = 0$ and $\mathbb{E}\left(|Z_i|^p\right) \leq 2^p\mathbb{E}\left(|Y_i|^p 1_{\{|Y_i|\geq\frac{3b_n}{t}\}}\right) \leq 2^p\mathbb{E}\left(|Y_i|^p\right) < \infty$, Lemma 1 applied to the independent variables $(Z_i)_{i\in\mathbb{N}^*}$ gives

$$\mathbb{E}\left(\left|\sum_{i=1}^{n} Z_i\right|^p\right) \leq c_p \max\left(\sum_{i=1}^{n}\mathbb{E}\left(|Z_i|^p\right), \left(\sum_{i=1}^{n}\mathbb{E}\left(Z_i^2\right)\right)^{p/2}\right). \tag{3}$$

It follows from (2) and (3) that

$$\begin{aligned}
U &\leq 2^p t^{-p} c_p \max\left(\sum_{i=1}^{n}\mathbb{E}\left(|Z_i|^p\right), \left(\sum_{i=1}^{n}\mathbb{E}\left(Z_i^2\right)\right)\right)\\
&\leq 2^{2p} t^{-p} c_p \max\left(\sum_{i=1}^{n}\mathbb{E}\left(|Y_i|^p 1_{\{|Y_i|\geq\frac{3b_n}{t}\}}\right), \left(\sum_{i=1}^{n}\mathbb{E}\left(Y_i^2 1_{\{|Y_i|\geq\frac{3b_n}{t}\}}\right)\right)^{p/2}\right)\\
&= C_p t^{-p}\max\left(r_{n,p}(t), (r_{n,2}(t))^{p/2}\right),
\end{aligned} \tag{4}$$

where $C_p = 2^{2p}c_p$.

The upper bound for V. Let us present one of the Bernstein inequalities. See, for instance, [6].

LEMMA 2 (Bernstein's inequality). Let $(X_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables such that, for any $n \in \mathbb{N}^*$ and any $i \in \{1,\dots,n\}$, $\mathbb{E}(X_i) = 0$ and $|X_i| \leq M < \infty$. Then, for any $\lambda > 0$ and any $n \in \mathbb{N}^*$, we have

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq \lambda\right) \leq \exp\left(-\frac{\lambda^2}{2(d_n^2 + \frac{\lambda M}{3})}\right),$$

where $d_n^2 = \sum_{i=1}^{n}\mathbb{E}(X_i^2)$.

For any $i \in \{1,\dots,n\}$, set $Z_i = Y_i 1_{\{|Y_i|<\frac{3b_n}{t}\}} - \mathbb{E}\left(Y_i 1_{\{|Y_i|<\frac{3b_n}{t}\}}\right)$. Since $\mathbb{E}(Z_i) = 0$ and $|Z_i| \leq |Y_i| 1_{\{|Y_i|<\frac{3b_n}{t}\}} + \mathbb{E}\left(|Y_i| 1_{\{|Y_i|<\frac{3b_n}{t}\}}\right) \leq \frac{6b_n}{t}$, Lemma 2 applied with the independent variables $(Z_i)_{i\in\mathbb{N}^*}$ and the parameters $\lambda = \frac{t}{2}$ and $M = \frac{6b_n}{t}$, gives

$$V \leq \exp\left(-\frac{t^2}{8\left(\sum_{i=1}^n \mathbb{V}\left(Y_i 1_{\left\{|Y_i|<\frac{3b_n}{t}\right\}}\right) + \frac{t}{6}\left(\frac{6b_n}{t}\right)\right)}\right).$$

Since $\sum_{i=1}^n \mathbb{V}\left(Y_i 1_{\left\{|Y_i|<\frac{3b_n}{t}\right\}}\right) \leq \sum_{i=1}^n \mathbb{E}\left(Y_i^2\right) = b_n$, it comes

$$V \leq \exp\left(-\frac{t^2}{16b_n}\right). \tag{5}$$

Putting (4) and (5) together, we obtain the inequality

$$\mathbb{P}\left(\sum_{i=1}^n Y_i \geq t\right) \leq U + V \leq C_p t^{-p} \max\left(r_{n,p}(t), (r_{n,2}(t))^{p/2}\right) + \exp\left(-\frac{t^2}{16b_n}\right).$$

Theorem 1 is proved.

Theorem 1 can be applied for a wide class of random variables. However, if the variables are almost surely absolutely bounded, or have finite moments of all orders satisfying some inequalities, the Bernstein inequalities can give more optimal results than (1). When it is hard or not possible to prove that these conditions are satisfied, Theorem 1 becomes of interest. This is illustrated in the example below and in Section 3 for the symmetric Pareto distribution. Other examples can be studied in a similar fashion.

EXAMPLE. Let $(X_i)_{i\in\mathbb{N}^*}$ be i.i.d. random variables such that, for any $x > 0$, $\mathbb{P}(|X_1| \geq x) \leq ce^{-x^\gamma}$, $\gamma > 0$, $c > 0$. Taking

$$t = t_n = (\kappa 16 \mathbb{E}(X_1^2) n \log n)^{1/2}$$

with $\kappa > 0$, the inequality (1) implies the existence of a constant $F_p > 0$ such that, for n and p large enough,

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^n X_i \geq t_n\right) &\leq F_p t_n^{-p} n \left(\mathbb{P}(|X_1| \geq \frac{3b_n}{t_n})\right)^{1/2} + \exp\left(-\frac{t_n^2}{16b_n}\right) \\ &\leq n^{-p/2+1} e^{-\frac{1}{2}\left(\frac{3\left(n\mathbb{E}(X_1^2)\right)^{1/2}}{(16\kappa \log n)^{1/2}}\right)^\gamma} + n^{-\kappa} \leq 2n^{-\kappa}.\end{aligned}$$

We used the inequality: $r_{n,u}(t) \leq n\left(\mathbb{E}\left(|X_1|^{2u}\right)\right)^{1/2}\left(\mathbb{P}(|X_1| \geq \frac{3b_n}{t_n})\right)^{1/2}$, $u \in \{1,2\}$.

3 Application to the Pareto Distribution

Proposition 1 below investigates the bound of the tail probability for a sum of n weighted i.i.d. random variables having the symmetric Pareto distribution.

PROPOSITION 1. Let $s > 2$ and $(X_i)_{i\in\mathbb{N}^*}$ be i.i.d. random variables with the probability density function

$$f(x) = \begin{cases} 2^{-1}s|x|^{-s-1}, & \text{if } |x| \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Let $(a_i)_{i\in\mathbb{N}^*}$ be a sequence of nonzero real numbers such that $\sum_{i=1}^{n}|a_i|^s < \infty$. Then, for any $n \in \mathbb{N}^*$, any $t \in \left(0, \frac{3b_n}{\rho_n}\right)$ with $\rho_n = \left(\sum_{i=1}^{n}|a_i|^s\right)^{1/s}$ and any $p \in \left(\max(\frac{s}{2}, 2), s\right)$, we have

$$\mathbb{P}\left(\sum_{i=1}^{n} a_i X_i \geq t\right) \leq K_p \left(t^{-2p+s} b_n^{p-s}\right) \sum_{i=1}^{n}|a_i|^s + \exp\left(-\frac{t^2}{16b_n}\right), \tag{7}$$

where $b_n = \frac{s}{s-2}\sum_{i=1}^{n} a_i^2$, $K_p = 3^{p-s}\max\left(\frac{s}{s-p}, \left(\frac{s}{s-2}\right)^{p/2}\right)2^{2p}c_p$,

$$c_p = \begin{cases} 1 + 2^{p/2}\pi^{-1/2}\Gamma\left(\dfrac{p+1}{2}\right), & 2 < p < 4, \\ \mathbb{E}\left(|\theta_1 - \theta_2|^p\right), & p \geq 4, \end{cases}$$

$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$, $a > 1$, and θ_1, θ_2 are independent Poisson random variables with parameter 2^{-1}.

PROOF. Let $n \in \mathbb{N}^*$. Set, for any $i \in \{1, \ldots, n\}$, $Y_i = a_i X_i$. Clearly, $(Y_i)_{i\in\mathbb{N}}$ are independent random variables such that, for any $i \in \{1, \ldots, n\}$, $\mathbb{E}(Y_i) = a_i\mathbb{E}(X_i) = 0$ and, for any $p \in \left(\max(\frac{s}{2}, 2), s\right)$, $\mathbb{E}(|Y_i|^p) \leq \sup_{i=1,\ldots,n}|a_i|^p\mathbb{E}(|X_i|^p) < \infty$. We are now in the position to apply Theorem 1. We have $b_n = \sum_{i=1}^{n}\mathbb{E}(Y_i^2) = \frac{s}{s-2}\sum_{i=1}^{n} a_i^2$. For any $u \in \{2, p\}$ and any $p \in \left(\max(\frac{s}{2}, 2), s\right)$, let us investigate the bound for $r_{n,u}(t) = \sum_{i=1}^{n}\mathbb{E}\left(|Y_i|^u 1_{\left\{|Y_i|\geq\frac{3b_n}{t}\right\}}\right) = \sum_{i=1}^{n}|a_i|^u\mathbb{E}\left(|X_i|^u 1_{\left\{|X_i|\geq\frac{3b_n}{|a_i|t}\right\}}\right)$. Recall that $\rho_n = \left(\sum_{i=1}^{n}|a_i|^s\right)^{1/s}$ and $\sigma_n = \sup_{i=1,\ldots,n}|a_i|$. Since $t \in \left(0, \frac{3b_n}{\rho_n}\right) \subseteq \left(0, \frac{3b_n}{\sigma_n}\right)$, for any $i \in \{1, \ldots, n\}$, we have $\mathbb{E}\left(|X_i|^u 1_{\left\{|X_i|\geq\frac{3b_n}{|a_i|t}\right\}}\right) = s\int_{\frac{3b_n}{|a_i|t}}^{\infty} x^{u-s-1}dx = \frac{s}{s-u}\left(\frac{3b_n}{|a_i|t}\right)^{u-s}$. Therefore,

$$r_{n,u}(t) = \frac{s}{s-u}\left(\frac{3b_n}{t}\right)^{u-s}\sum_{i=1}^{n}|a_i|^s$$

and

$$\max\left(r_{n,p}(t), (r_{n,2}(t))^{p/2}\right) \leq R_p\left(\frac{3b_n}{t}\right)^p \max\left(\left(\frac{3b_n}{t\rho_n}\right)^{-s}, \left(\left(\frac{3b_n}{t\rho_n}\right)^{-s}\right)^{p/2}\right),$$

where $R_p = \max\left(\frac{s}{s-p}, \left(\frac{s}{s-2}\right)^{p/2}\right)$. Since $t \in \left(0, \frac{3b_n}{\rho_n}\right)$ and $p > 2$, we have

$$\max\left(\left(\frac{3b_n}{t\rho_n}\right)^{-s}, \left(\left(\frac{3b_n}{t\rho_n}\right)^{-s}\right)^{p/2}\right) = \left(\frac{3b_n}{t\rho_n}\right)^{-s}.$$

Hence,

$$\max\left(r_{n,p}(t),(r_{n,2}(t))^{p/2}\right) \leq R_p\left(\frac{3b_n}{t}\right)^{p-s}\sum_{i=1}^{n}|a_i|^s. \tag{8}$$

Theorem 1 and (8) imply that

$$\mathbb{P}\left(\sum_{i=1}^{n}a_iX_i \geq t\right) \leq K_p\left(t^{-2p+s}b_n^{p-s}\right)\sum_{i=1}^{n}|a_i|^s + \exp\left(-\frac{t^2}{16b_n}\right),$$

where $b_n = \frac{s}{s-2}\sum_{i=1}^{n}a_i^2$, $K_p = 3^{p-s}\max\left(\frac{s}{s-p},\left(\frac{s}{s-2}\right)^{p/2}\right)C_p$, and $C_p = 2^{2p}c_p$. Using the optimal form of the Rosenthal constant c_p for the symmetric random variables (see [1], [4] and [10]), we complete the proof of Proposition 1.

For recent asymptotic results on the approximation of the tail probability of a sum of n i.i.d. random variables having the symmetric Pareto distribution, we refer to [3].

Section 4 below compares the precision between (7) and the bound obtained via the Fuk-Nagaev inequality.

4 Comparison with the Fuk-Nagaev Inequality

For the sake of clarity, recall the Fuk-Nagaev inequality. The considered version can be found in [5].

LEMMA 3. (Fuk-Nagaev inequality) Let $p \geq 2$ and $(Y_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables such that, for any $n \in \mathbb{N}^*$ and any $i \in \{1,\ldots,n\}$, $\mathbb{E}(Y_i) = 0$ and $\mathbb{E}(Y_i^2) < \infty$. Then, for any $t > 0$, we have

$$\begin{aligned}\mathbb{P}&\left(\sum_{i=1}^{n}Y_i \geq t\right)\\ &\leq \sum_{i=1}^{n}\mathbb{P}(Y_i \geq \eta t) + (\eta t)^{-p}\sum_{i=1}^{n}\mathbb{E}\left(Y_i^p 1_{\{0\leq Y_i\leq \eta t\}}\right) + \exp\left(-\frac{t^2}{c_p^* d_n}\right),\end{aligned} \tag{9}$$

where $d_n = \sum_{i=1}^{n}\mathbb{E}(Y_i^2)$, $\eta = \frac{p}{p+2}$, and $c_p^* = \frac{e^p(p+2)^2}{2}$.

In some cases, (7) can give better results than the Fuk-Nagaev inequality. For instance, consider the symmetric Pareto distribution (i.e. $(X_i)_{i\in\mathbb{N}^*}$ are i.i.d. with the probability density function (6) with $s > 2$). Suppose that n is large. For any $p \in \left(\max(\frac{s}{2},2),s\right)$, if we take $t = t_n = 2^{3/2}(sn\log n)^{1/2}$ $(\in \left(0,3n^{1-1/s}\right))$, then we can balance the two terms of the bound in (7); there exists a constant $Q_p > 0$ such that

$$\mathbb{P}\left(\sum_{i=1}^{n}X_i \geq t_n\right) \leq Q_p n^{1-s/2}(\log n)^{-p+s/2} + n^{1-s/2} \leq 2n^{1-s/2}. \tag{10}$$

For the same t_n, the Fuk-Nagaev inequality (9) implies the existence of a constant $R_p > 0$ such that, for any $p > 0$,

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq t_n\right) &\leq R_p n^{1-p/2} (\log n)^{-p/2} + n^{\frac{16}{c_p^*}(1-s/2)} \\ &\leq 2\max\left(n^{1-p/2}, n^{\frac{16}{c_p^*}(1-s/2)}\right). \end{aligned} \tag{11}$$

Since $c_p^* = \frac{e^p(p+2)^2}{2} > 8e^2 > 16$, for n large enough, the rate of convergence in (10) is faster than the one in (11). Therefore, in this case, (7) gives a better result than the Fuk-Nagaev inequality. This superiority can be extended to $t_n = \kappa(n \log n)^{1/2}$ for $0 < \kappa < 2^{3/2} s^{1/2}$.

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