Applied Mathematics E-Notes, 7(2007), 23-31 © ISSN 1607-2510
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On Spectral Properties Of The Laplace Operator Via Boundary Perturbation*

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Received 26 August 2006

Abstract

We investigate the interplay between the geometry, boundary conditions and spectral properties of the Laplace operator under deformation of the domain in three-dimensional space $\mathbb{R}^3$. Using integral equations, we show that the eigenfunctions are jointly holomorphic functions in the spatial and boundary-variation variables. Some results for the convergence estimates are established.

1 Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^3$, with a smooth boundary $\partial\Omega$. We suppose that this boundary $\partial\Omega$ is parameterized by the function: $\gamma(s,t) : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3$ which is analytic, π-periodic in the variable s, and 2π-periodic in the variable t. Let us consider the following eigenvalue problem for the Laplace operator in the domain Ω of the three-dimensional space $\mathbb{R}^3$:

$$-\Delta u_0(x) = \lambda_0^2 u_0(x), \quad x \in \Omega, \quad \text{and } u_0(x) = 0, \quad x \in \partial\Omega. \tag{1}$$

Throughout this paper, the domain Ω is supposed to be perturbed according to some parameter ϵ and therefore the problem (1) is transformed into the problem (2) as described in Section 2. The properties of eigenvalue problems under shape deformation have been the subject of comprehensive studies [5, 11] and the area continues to carry great importance to this day [1, 3, 4, 7, 8, 10, 9, 12]. A substantial portion of these investigations relate to properties of smoothness and analyticity of eigenvalues and eigenfunctions with respect to perturbations. Bruno and Reitich have presented in [3, Theorem 2, p.172 and Section 3, pp.180-183] some explicit constructions of high-order boundary perturbation expansions for eigenelements in two dimensions. Their algorithm is based on certain properties of joint analytic dependence on the boundary perturbations and spatial variables of the eigenfunctions. The paper is organized as follows. Section 2 provides the formulation of the main problem in this paper and contains the application of the integral equations method to the Dirichlet eigenvalue problem for the Laplace operator. In particular, we rigorously establish the existence

*Mathematics Subject Classifications: 35P05, 41A60, 47A55.

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of an operator-valued function L_ϵ and we establish that this operator define complex analytic functions of the spatial variable x and the height parameter ϵ. Finally, in section 3, in Theorem 3.1 we show that the eigenfunctions $u_j(\epsilon)$ of problem (2) are jointly analytic in (x, ϵ) and satisfy an uniform asymptotic expansion. We close this section by showing and proving some results concerning the convergence estimates of the eigenfunctions with respect to the parameter ϵ.

2 Problem Description

We start by introducing the analytic function $\beta : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$, $(s, t) \mapsto \beta(s, t)$ to be π-periodic in the variable s and 2π-periodic in the variable t. Let

$$\gamma_\epsilon(s, t) = \gamma(s, t) + \epsilon\beta(s, t), \quad \epsilon \in \mathbb{R}.$$

With this definition, $(s, t; \epsilon) \mapsto \gamma_\epsilon(s, t)$ is an analytic function on $[0, \pi] \times [0, 2\pi] \times \mathbb{R}$, π-periodic in the variable s, 2π-periodic in the variable t. Now we consider the bounded domain Ω_ϵ in $\mathbb{R}^3$ with boundary $\partial\Omega_\epsilon$ parameterized by the function $\gamma_\epsilon(s, t)$:

$$\partial\Omega_\epsilon = \{\gamma_\epsilon(s, t); \quad (s, t) \in [0, \pi] \times [0, 2\pi]\}.$$

The outward unit normal to $\partial\Omega_\epsilon$ is denoted by ν_ϵ and for $\epsilon = 0$ we naturally write down $\Omega_0 \equiv \Omega$.

In this paper, we deal with the asymptotic of eigenvalues and eigenfunctions associated with the following eigenvalue problem:

$$\begin{cases} -\Delta u(\epsilon) = \lambda^2(\epsilon) u(\epsilon) & \text{in } \Omega_\epsilon, \\ u(\epsilon) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{2}$$

It is well known that the operator $-\Delta$ on $L^2(\Omega_\epsilon)$ with domain $H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ is self-adjoint with compact resolvent. Consequently, its spectrum consists entirely of isolated, real and positive eigenvalues with finite multiplicity, and there are corresponding eigenfunctions which make up an orthonormal basis of $L^2(\Omega_\epsilon)$. Throughout this paper, we denote by $\| \cdot \|$ the norm associated to the scalar product $< \cdot, \cdot >$ on $L^2(\Omega)$. Let $\lambda_0^2 > 0$ denote an eigenvalue of the eigenvalue problem (1) for $\epsilon = 0$ with geometric multiplicity $m \geq 1$. There exists a small constant $r_0 > 0$ such that λ_0^2 is the unique eigenvalue of (2) for $\epsilon = 0$ in the set $\left\{\lambda^2, \lambda \in D_{r_0}(\lambda_0)\right\}$, where $D_{r_0}(\lambda_0)$ is a disc of center λ_0 and radius r_0. Let us call the λ_0-group *the totality of the perturbed eigenvalues of* (2) for $\epsilon > 0$ generated by "splitting" from λ_0^2.

We now develop a boundary integral formulation for solving the eigenvalue problem (2). Throughout this paper, we use for simplicity the notation $H_\sharp^\varrho(]0, \pi[\times]0, 2\pi[) = H^\varrho(\mathbb{R}^2/]0, \pi[\times]0, 2\pi[)$, for $\varrho \in \mathbb{R}$, where $H^\varrho(\mathbb{R}^2/]0, \pi[\times]0, 2\pi[)$ denotes the classical Sobolev H^ϱ-space on the quotient $\mathbb{R}^2/]0, \pi[\times]0, 2\pi[$. Using change of variables and integral equations, the following result immediately holds (see [1, 13]).

PROPOSITION 1. Let $L_\epsilon(\lambda) : H_\sharp^{-1/2}(]0, \pi[\times]0, 2\pi[) \to H_\sharp^{1/2}(]0, \pi[\times]0, 2\pi[)$ be defined as follows: for $f \in H_\sharp^{-1/2}(]0,\pi[\times]0,2\pi[)$,

$$\begin{aligned} L_\epsilon(\lambda)f(s,t) &= (S(\lambda)f(\gamma_\epsilon^{-1}))(\gamma_\epsilon(s,t)) \\ &= \int_0^{2\pi}\int_0^{\pi} G(\gamma_\epsilon(s,t),\gamma_\epsilon(s',t'))|\nabla\gamma_\epsilon(s',t')|f(s',t')ds'dt'. \end{aligned}$$

Then the operator-valued function $L_\epsilon(\lambda)$ is Fredholm analytic with index 0 in $\mathbb{C}\setminus i\mathbb{R}^-$. Moreover, $L_\epsilon^{-1}(\lambda)$ is a meromorphic function and its poles are in $\{\Im(z)\le 0\}$, where $\Im(z)$ means the imaginary part of z and $\Re(z)$ is the real part.

LEMMA 1. There exist positive numbers ϵ_1, ρ,η, and r_0 such that the kernel of the operator $L_\epsilon(\lambda)$ has the form:

$$\begin{aligned} G(\gamma_\epsilon(s,t),\gamma_\epsilon(s',t'))|\nabla\gamma_\epsilon(s',t')| &= g(s,t,s',t'\epsilon,\lambda) \\ &+\sum_{p=-1}^{1}\sum_{k=-1}^{1}\frac{h(s,t,s'+k\pi,t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-(s'+k\pi))^2}}, \end{aligned}$$

for $(s,t,s',t',\epsilon,\lambda)\in\mathcal{J}$, where $h(s,t,s',t',\epsilon,\lambda)$ and $g(s,t,s',t',\epsilon,\lambda)$ are analytic with respect to $(s,t,s',t',\epsilon,\lambda)$ in $\mathcal{J}$. Here we have put $\mathcal{J}=\{|\Im(s)|\le\eta,|\Im(t)|\le\eta;|\Im(s')|\le\eta,|\Im(t')|\le\eta;|\epsilon|\le\epsilon_1;\lambda\in D_{r_0}(\lambda_0);-\rho\le\Re(s),\Re(s')\le\pi+\rho;-\rho\le\Re(t),\ \Re(t')\le 2\pi+\rho\}$.

PROOF. Upon replacing x by $\gamma_\epsilon(s,t)$ and x' by $\gamma_\epsilon(s',t')$, we immediately obtain the following result for the kernel of L_ϵ, provided ϵ_1, ρ and η are sufficiently small:

$$\frac{1}{4\pi}\frac{e^{i\lambda|\gamma_\epsilon(s,t)-\gamma_\epsilon(s',t')|}}{|\gamma_\epsilon(s,t)-\gamma_\epsilon(s',t')|}=\frac{h(s,t,s',t',\epsilon,\lambda)}{((s-s')^2+(t-t')^2)^{1/2}},$$

where h is a function defined in the set $\mathcal{J}$. In fact, we have

$$h(s,t,s',t',\epsilon,\lambda)=G(\gamma_\epsilon(s,t),\gamma_\epsilon(s',t'))|\nabla\gamma_\epsilon(s',t')|((s-s')^2+(t-t')^2)^{1/2}.$$

Using classical results, and the fact γ_ϵ is analytic, we see that the function h and its derivatives are analytic in the set $\mathcal{J}$.

To proceed to the proof, we use some idea little close to that found in the proof of Theorem 6.1 in [2, pp. 331-333]. The fact that γ_ϵ is π-periodic in the variable s' and 2π-periodic in the variable t', there exists a function $g(s,t,s',t',\epsilon,\lambda)$ such that

$$\begin{aligned} \frac{h(s,t,s',t',\epsilon,\lambda)}{((s-s')^2+(t-t')^2)^{1/2}} &= \sum_{p=-1}^{1}\sum_{k=-1}^{1}\frac{h(s,t,s'+k\pi,t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-(s'+k\pi))^2}} \\ &+g(s,t,s',t',\epsilon,\lambda), \end{aligned}$$

where this function g is given by:

$$\begin{aligned} g(s,t,s',t',\epsilon,\lambda) &= -\Big[\sum_{k=-1}^{1}\frac{h(s,t,s'+k\pi,t'-2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'-2\pi))^2+(s-(s'+k\pi))^2}} \\ &+\sum_{k=-1}^{1}\frac{h(s,t,s'+k\pi,t'+2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2\pi))^2+(s-(s'+k\pi))^2}}\Big]. \end{aligned} \tag{3}$$

The analyticity of the function g follows, evidently, from that of h.

Considering the results and notations established in Lemma 1 the following holds.

THEOREM 1. There exists a constant $\eta > 0$ and a complex neighborhood $\mathcal{V}$ of 0 such that for every function $\phi(s,t;\epsilon) \in H_\sharp^{-1/2}(]0,\pi[\times]0,2\pi[)$ analytic in $(s,t;\epsilon) \in \{|\Im(s)|, |\Im(t)| \leq \eta\} \times \mathcal{V}$, the function $L_\epsilon(\lambda)\phi(s,t;\epsilon) \in H_\sharp^{1/2}(]0,\pi[\times]0,2\pi[)$ is analytic with respect to $(s,t;\epsilon,\lambda) \in \{|\Im(s)|, |\Im(t)| \leq \eta\} \times \mathcal{V} \times D_{r_0}(\lambda_0)$ where $D_{r_0}(\lambda_0)$ is a disc of center λ_0 and radius r_0.

PROOF. There is a central difficulty to prove the analytic property of the operator L_ϵ. This difficulty appears from the spatial singularity of its kernel. To establish this regularity we may focus , for simplicity, our attention to the change of variables when we integrate by parts as done in [2, Lemma 6.2]. According to Lemma 1, there exist functions F and $\mathcal{G}$ such that,

$$L_\epsilon(\lambda)f(s,t) = F(s,t,\epsilon,\lambda) + \mathcal{G}(s,t,\epsilon,\lambda),$$

where the function F is given by:

$$F(s,t,\epsilon,\lambda) = \sum_{p=-1}^{1} \int_0^{2\pi} \int_0^{\pi} \sum_{k=-1}^{1} \frac{h(s,t,s'+k\pi,t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2 + (s-(s'+k\pi))^2}} f(s',t')ds'dt'.$$

Next, relation (3) implies that the analyticity of $\mathcal{G}(s,t,\epsilon,\lambda)$ is deduced from that of the following function:

$$(s,t,\epsilon,\lambda) \mapsto \sum_{k=-1}^{1} \int_0^{2\pi} \int_0^{\pi} \frac{h(s,t,s'+k\pi,t'\pm 2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'\pm 2\pi))^2 + (s-(s'+k\pi))^2}} ds'dt'.$$

To do this, we introduce the function

$$G_1(s,t,t',\epsilon,\lambda) = \sum_{k=-1}^{1} \int_0^{\pi} \frac{h(s,t,s'+k\pi,t'+2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2\pi))^2 + (s-(s'+k\pi))^2}} ds',$$

and by a change of variables, we get

$$G_1(s,t,t',\epsilon,\lambda) = \int_{-\pi}^{2\pi} \frac{h(s,t,s',t'+2\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2\pi))^2 + (s-s')^2}} ds'.$$

Further, if we define

$$K_1(s,t,s',t',\epsilon,\lambda) = \int_s^{s'} h(s,t,z,t'+2\pi,\epsilon,\lambda)dz,$$

integration by parts yields

$$G_1(s,t,t',\epsilon,\lambda) = \big((t-t'-2\pi)^2 + (s-2\pi)^2\big)^{-1/2} K_1(s,t,2\pi,t',\epsilon,\lambda)$$
$$-\big((t-t'-2\pi)^2 + (s+\pi)^2\big)^{-1/2} K_1(s,t,-\pi,t',\epsilon,\lambda)$$

$$+\int_{-\pi}^{2\pi}\left((t-t'-2\pi)^2+(s-s')^2\right)^{-3/2}K_1(s,t,s',t',\epsilon,\lambda)ds'.$$

Clearly, the function $(s,t,\epsilon,\lambda)\to\int_0^{2\pi}G_1(s,t,t',\epsilon,\lambda)dt'$ can be extended to a complex analytic function in $\mathbb{C}\times\mathbb{C}\times\mathcal{V}\times D_{r_0}(\lambda_0)$ and so the analyticity of $\mathcal{G}(s,t,\epsilon,\lambda)$ holds. We now prove the result for the function $F(s,t,\epsilon,\lambda)$. As was done for the proof of $\mathcal{G}$, we first remark that

$$\begin{aligned}&\sum_{k=-1}^{1}\int_0^{\pi}\frac{h(s,t,s'+k\pi,t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-(s'+k\pi))^2}}ds'\\ =\;&\int_{-\pi}^{2\pi}\frac{h(s,t,s',t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-s')^2}}ds'.\end{aligned}$$

Therefore,

$$F(s,t,\epsilon,\lambda)=\sum_{p=-1}^{1}\int_0^{2\pi}\int_{-\pi}^{2\pi}\frac{h(s,t,s',t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-s')^2}}ds'dt'.$$

In other words, by change of variables we have

$$\sum_{p=-1}^{1}\int_0^{2\pi}\frac{h(s,t,s',t'+2p\pi,\epsilon,\lambda)}{\sqrt{(t-(t'+2p\pi))^2+(s-s')^2}}dt'=\int_{-2\pi}^{4\pi}\frac{h(s,t,s',t',\epsilon,\lambda)}{\sqrt{(t-t')^2+(s-s')^2}}dt'.$$

Hence, the following relation is valid

$$F(s,t,\epsilon,\lambda)=\int_{-2\pi}^{4\pi}\int_{-\pi}^{2\pi}\frac{h(s,t,s',t',\epsilon,\lambda)}{\sqrt{(t-t')^2+(s-s')^2}}ds'dt',$$

and can be extended to a complex analytic function in $\mathbb{C}\times\mathbb{C}\times\mathcal{V}\times D_{r_0}(\lambda_0)$.

3 Asymptotic Formula

In this section, we develop the asymptotic expansions of the eigenfunctions of (2) when the parameter ϵ goes to zero. Next, we give the following Lemma to prove the main result in this section. Its proof is not difficult if one remark that the surface $\partial\Omega_\epsilon$ is non characteristic for Δ_x and the Cauchy-Kowaleski theorem immediately give the result (see [1] and [6, Lemma 5.3]) for more details.

LEMMA 2. The functions given by $\hat{u}_{i,j}(\epsilon)(x)=L_\epsilon(\lambda_j(\epsilon))\partial_\nu u_i(\epsilon)(\gamma^{-1})$ are jointly analytic in the variables $(x,\epsilon)\in\mathcal{K}_0\times]-\epsilon_1,\epsilon_1[$, where $\mathcal{K}_0$ is a bounded neighborhood of $\overline{\Omega}_0$ in $\mathbb{R}^3$.

We summarize the main result as follows.

THEOREM 2. Let $\mathcal{K}_0$ be a bounded neighborhood of $\overline{\Omega}_0$ in $\mathbb{R}^3$. Then there exists a constant $\epsilon_2=\epsilon_2(\epsilon_1)>0$ smaller than ϵ_1 such that an orthonormal basis of eigenfunctions $(u_j(\epsilon))_j$ corresponding to the λ_0-group, $(\lambda_j^2(\epsilon))_j$, in $H_0^1(\Omega_\epsilon)$ can be chosen to depend holomorphically in $(x, \epsilon) \in \mathcal{K}_0 \times]-\epsilon_2, \epsilon_2[$. Moreover these eigenfunctions satisfy the following asymptotic formulae

$$u_j(\epsilon) = u_0^{(j)} + \sum_{n \geq 1} u_n^{(j)} \epsilon^n.$$

The family $u_0^{(j)}$ builds a basis of eigenfunctions of (1) associated to λ_0^2 and normalized in $L^2(\Omega_0)$.

PROOF. Let $\partial_{\nu_\epsilon} u_j(\epsilon)(\gamma_\epsilon(s,t)) = q_j(\epsilon)(s,t)$. The family of functions $(q_j(\epsilon))_{1 \leq j \leq m}(s,t)$ (where $q_j(\epsilon) \in H_\sharp^{-1/2}(]0,\pi[\times]0,2\pi[)$) builds an orthonormal basis of $Ker(L_\epsilon(\lambda_j(\epsilon)))$ (see [1, 6]), which is analytic in $\mathbb{R}^2 \times]-\epsilon_1, \epsilon_1[$. In other words, $L_\epsilon(\lambda_j(\epsilon))\partial_\nu u_i(\epsilon)(\gamma^{-1})$ forms a basis of eigenfunctions of the eigenvalue problem (2) associated to $\lambda_j^2(\epsilon)$. Using the Schmidt orthogonalization process, we construct the desired orthonormal basis. Clearly, the functions $(\hat{u}_{i,j}(\epsilon))_{ij}$, introduced in Lemma 2, build a basis of the eigenspaces corresponding to the λ_0-group, $(\lambda_j^2(\epsilon))_j$ in $H_0^1(\Omega_\epsilon)$. We will now give the asymptotic expansion of these functions when ϵ tends to 0. To simplify notations we drop the subscripts i and j. Integral equations give

$$\hat{u}(\epsilon)(x) = \int_0^{2\pi} \int_0^\pi G(x, \gamma_\epsilon(s,t)) q(\epsilon)(s,t) |\nabla \gamma_\epsilon(s,t)| ds dt. \tag{4}$$

The perturbed eigenvalue $\lambda(\epsilon)$ lies in a small neighborhood of λ_0 for small values of ϵ. Then, there exists $\epsilon_2 > 0$ ($\epsilon_2 \leq \epsilon_1$), such that we have the following Taylor expansion

$$G(x, \gamma_\epsilon(s,t)) |\nabla \gamma_\epsilon(s,t)| = G(x, \gamma(s,t)) |\nabla \gamma(s,t)| + \sum_{k \geq 1} \epsilon^k G_k(x, \gamma(s,t); \lambda), \tag{5}$$

which holds uniformly in $x \in \bar{\mathcal{K}}_0$ and $(s,t) \in [0,\pi] \times [0, 2\pi]$. Using Theorem 1 we write

$$q(\epsilon)(s,t) = q_0(s,t) + \sum_{k \geq 1} \epsilon^k q_k(s,t), \tag{6}$$

uniformly in $(s,t) \in [0,\pi] \times [0,2\pi]$. Substituting the last two asymptotics into (4) we find

$$\hat{u}(\epsilon) = \hat{u}(0) + \sum_{k \geq 1} \epsilon^k [\sum_{n=1}^{k} \int_0^{2\pi} \int_0^\pi q_{k-n}(s,t) G_n(x, \gamma(s,t); \lambda) ds dt]. \tag{7}$$

Now we use the Schmidt orthogonalization process to construct from the eigenfunctions $(\hat{u}_j(\epsilon))_j$ an orthonormal basis $(u_j(\epsilon))_j$.

LEMMA 3. There exist some positive constants ϵ_3 and C, such that

$$\|\nabla(u - u_0)\|_{L^2(\Omega_\epsilon)} \leq C |\Omega_\epsilon \backslash \Omega_0|^{1/2},$$

for $0 < \epsilon < \epsilon_3$. The constant C depends on (λ_0, u_0), but is otherwise independent of ϵ.

PROOF. From definition of Ω_ϵ and without loss of generality, we can easily found $\alpha_1 > 0$ such that $\Omega_0 \subset \Omega_\epsilon$ and $\partial\Omega_\epsilon \cap \partial\Omega_0 = \emptyset$, for $0 < \epsilon < \alpha_1$.

Define the open bounded domain $\tilde{\Omega}_\epsilon = \Omega_\epsilon \backslash \overline{\Omega}_0$ and the function $w(\epsilon) = u(\epsilon) - u_0$, for $0 < \epsilon < \inf(\epsilon_2, \alpha_1)$ where ϵ_2 is given by Theorem 2. Using the equations (1) and (2), we compute that $w(\epsilon)$ solves:

$$-\Delta w = \lambda^2 w + (\lambda^2 - \lambda_0^2) u_0 \text{ in } \tilde{\Omega}_\epsilon. \tag{8}$$

For $z \in \mathbb{R}$, we define the function ϑ by

$$\vartheta(z) = \lambda^2 z + (\lambda^2 - \lambda_0^2) \|u_0\|_{L^\infty(\Omega_0)}.$$

Then, we trivially have

$$|\vartheta(w(\epsilon))| \leq |\vartheta(0)| + \lambda^2 |w(\epsilon)|. \tag{9}$$

Now it turns out from the definition of w_ϵ that $w_\epsilon \to 0$ as $\epsilon \to 0$ together with the fact that $\|u_0\|_{L^\infty(\Omega_0)} > 0$ justify that there exists $0 < \alpha_2 < \inf(\epsilon_2, \alpha_1)$ such that for $0 < \epsilon < \alpha_2$,

$$|w_\epsilon| \leq 2\|u_0\|_{L^\infty(\Omega_0)}, \text{ for } x \in \tilde{\Omega}_\epsilon.$$

Moreover, there exists $\alpha_3 \geq 0$ such that $\lambda^2 \leq \lambda_0^2 + \frac{1}{3}$, for $0 \leq \epsilon \leq \alpha_3$. Now, it is useful to introduce the following function:

$$\tilde{\vartheta}(w) = \lambda^2 w + (\lambda^2 - \lambda_0^2) u_0,$$

where w is the solution of (8). If we examine each term on the right hand side of (9) separately, we find out that the first term is bounded by

$$|\vartheta(0)| \leq \frac{1}{3} \|u_0\|_{L^\infty(\Omega_0)}, \text{ for } 0 < \epsilon < \alpha_3.$$

The second term is bounded by

$$\lambda^2 |w_\epsilon| \leq 2(\lambda_0^2 + \frac{1}{3}) \|u_0\|_{L^\infty(\Omega_0)}, \text{ for } 0 < \epsilon < \alpha_4 = \inf(\alpha_2, \alpha_3).$$

These estimates give

$$\|\tilde{\vartheta}(w_\epsilon)\|_{L^\infty(\Omega_\epsilon)} \leq (1 + 2\lambda_0^2) \|u_0\|_{L^\infty(\Omega_0)}, \text{ for } 0 < \epsilon < \alpha_4. \tag{10}$$

Next, the relation (8) implies

$$-\Delta w(\epsilon) = \tilde{\vartheta}(w(\epsilon)).$$

By integrating by parts and if we use relation (10), we find that the function $w(\epsilon)$ verifies:

$$\|\nabla w\|^2_{L^2(\tilde{\Omega}_\epsilon)} = \int_{\tilde{\Omega}_\epsilon} \tilde{\vartheta}(w) \bar{w} dx \leq (1 + 2\lambda_0^2) |\tilde{\Omega}_\epsilon|^{1/2} \|u_0\|_{L^\infty(\Omega_0)} \|w\|_{L^2(\tilde{\Omega}_\epsilon)}.$$

So, if we use the fact that $w(\epsilon)$ vanishes in Ω_0, then we have

$$\|\nabla w\|^2_{L^2(\Omega_\epsilon)} \leq (1 + 2\lambda_0^2) |\tilde{\Omega}_\epsilon|^{1/2} \|u_0\|_{L^\infty(\Omega_0)} \|w\|_{L^2(\Omega_\epsilon)}. \tag{11}$$

By Poincare's inequality, there exists some positive constant $C(\Omega_\epsilon)$ such that

$$\|w\|_{L^2(\Omega_\epsilon)} \le C(\Omega_\epsilon)\|\nabla w\|_{L^2(\Omega_\epsilon)}.$$

The fact w and ∇w are uniformly bounded on Ω_ϵ implies there exists some constant C_1 independent of ϵ (e.g.[7, p.33]) such that

$$C(\Omega_\epsilon) \le C_1, \tag{12}$$

and (11) becomes $\|\nabla w\|_{L^2(\Omega_\epsilon)} \le C_1(1+2\lambda_0^2)|\tilde{\Omega}_\epsilon|^{1/2}\|u_0\|_{L^\infty(\Omega_0)}$. The choice $\epsilon_3 = \alpha_4$ and $C = C_1(1+2\lambda_0^2)\|u_0\|_{L^\infty(\Omega_0)}$ concludes the proof.

Our main result in this section is the following.

THEOREM 3. Let γ, β and Ω_ϵ be defined as in Section 2. Then, there exist some constant $0 < \epsilon_4 \le 1/M$, $M = \max|\beta(s,t)|$ and some positive constant κ dependent on λ_0, u_0, $|\gamma|$ and M but otherwise independent of ϵ such that

$$\|u - u_0\|_{L^2(\Omega_\epsilon)} \le \kappa\epsilon^{1/2}, \quad \text{for } 0 < \epsilon < \epsilon_4.$$

PROOF. For simplicity we can suppose that Ω_0 is a ball with radius $\varrho_0 > 0$ in $\mathbb{R}^3$. It then follows that $|\gamma(s,t)| = \varrho_0$. It is not hard to see that, in spherical coordinates (ϱ, θ, ϕ), there exists a regular function $\Upsilon : [0,\pi] \times [0, 2\pi] \to \mathbb{R}_+$; $\Upsilon(\theta,\phi) = |\beta(\theta,\phi)|$ such that the boundary $\partial\Omega_\epsilon$ can be re-parameterized by

$$\varrho = \varrho(\epsilon,\theta,\phi) = \varrho_0 + \epsilon\Upsilon(\theta,\phi), \quad (\theta,\phi) \in [0,\pi] \times [0,2\pi].$$

Therefore

$$|\tilde{\Omega}_\epsilon| = \int_0^{2\pi}\int_0^{\pi}[\int_{\varrho_0}^{\varrho_0+\epsilon\Upsilon(\theta,\phi)} \varrho^2 \sin(\theta) d\varrho] d\theta d\phi = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}[\epsilon^3\Upsilon^3(\theta,\phi)$$
$$+3\epsilon\varrho_0^2\Upsilon(\theta,\phi) + 3\epsilon^2\varrho_0\Upsilon^2(\theta,\phi)]\sin(\theta)d\theta d\phi. \tag{13}$$

For $\epsilon < \epsilon_4 = \inf(\epsilon_3, 1/M)$, we can write: $\epsilon^2 \le \epsilon/M$. Then

$$\epsilon^i M^i \le \epsilon M \quad \text{for } i \in \{2,3\}.$$

Consequently the equality (13) gives

$$|\tilde{\Omega}_\epsilon| \le \frac{4\pi}{3}M(1 + 3\varrho_0 + 3\varrho_0^2)\epsilon. \tag{14}$$

By Poincare's inequality and Lemma 3 we write

$$\|u - u_0\|_{L^2(\Omega_\epsilon)} \le C(\Omega_\epsilon)C|\tilde{\Omega}_\epsilon|^{1/2}.$$

Finally, we obtain the desired result if we consider the relations (12) and (14) and if we choose the constant $\kappa = 2CC_1\sqrt{\frac{\pi}{3}M(1+3\varrho_0+3\varrho_0^2)}$.

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