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The Holditch Sickles For The Open Homothetic Motions*

Salim Yüce†, Nuri Kuruoğlu‡

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Abstract

A. Tutar and N. Kuruoğlu [1] had given the following theorem as a generalization of the classical Holditch Theorem [2]: During the closed planar homothetic motions with the period T, if the chord $\overline{AB}$ of fixed lenght $a+b$ is moved around once on an oval k_0, then a point $X \in \overline{AB}$ ($a = \overline{AX}, b = \overline{BX}$) describes a closed path $k_0(X)$ and the "Holditch Ring", which is bounded by k_0 and $k_0(X)$ has the surface area

$$F = h^2(t_0)\pi ab,$$

for $\exists\ t_0 \in [0, T]$. In this paper, under the open homothetic motions we expressed the Holditch Sickle such that the closed oval is replaced by the boundary of an bounded convex domain and so, the Holditch Sickles given by H. Pottmann [3] for one-parameter Euclidean motions generalized to the homothetic motions.

1 Introduction

Let E and E' be moving and fixed Euclidean planes and $\{O;\ \mathbf{e_1}, \mathbf{e_2}\}$ and $\{O';\ \mathbf{e'_1}, \mathbf{e'_2}\}$ be their coordinate systems, respectively. By taking $\mathbf{OO'} = \mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2}$, for $u_1, u_2 \in \mathbf{R}$, the motion defined by the transformation

$$\mathbf{x}' = h\mathbf{x} - \mathbf{u} \tag{1}$$

is called *one-parameter planar homothetic motion* with the *homothetic scale* $h(t)$ and denoted by $H_1 = E/E'$, where $\mathbf{x}$ and $\mathbf{x}'$ are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X \in E$, respectively. Furthermore, at the initial time $t = 0$ the coordinate systems are coincident. Taking $\varphi = \varphi(t)$ as the *rotation angle* between $\mathbf{e_1}$ and $\mathbf{e'_1}$, the equation

$$\begin{aligned} \mathbf{e_1} &= \quad\cos\varphi\,\mathbf{e'_1} + \sin\varphi\,\mathbf{e'_2} \\ \mathbf{e_2} &= -\sin\varphi\,\mathbf{e'_1} + \cos\varphi\,\mathbf{e'_2} \end{aligned} \tag{2}$$

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†Yıldız Technical University, Faculty of Arts and Science, Department of Mathematics, Esenler, 34210, İstanbul, Turkey

‡University of Bahçeşehir, Faculty of Arts and Science, Department of Mathematics and Computer Sciences, Beşiktaş 34100, İstanbul, Turkey

can be written. Also, the homothetic scale h, the rotation angle φ and u_1, u_2 are continuously differentiable functions of a real time parameter t. If

$$\begin{aligned} h(t+T) &= h(t), \quad \varphi(t+T) = \varphi(t) + 2\pi\nu, \quad \forall t \in [0,T], \\ & u_j(t+T) = u_j(t), \quad j = 1,2 \end{aligned}$$

then H_1 is called *one-parameter closed planar homothetic motion* with the period $T > 0$ and the *rotation number* $\nu \in \mathbf{Z}$. Otherwise, H_1 is called *one-parameter open planar homothetic motion*.

2 Holditch Sickles under Homothetic Motions

During the open homothetic motion H_1 , let unbounded convex curve k_o be the common orbit curve of the points A and B of the moving plane E and the points A and B move to infinity for $t \to \mp\infty$. Also, there could be a pair of different, parallel tangents t_1,t_2 of k_o, which is the edge of an unbounded convex domain $K_o \subset E'$. If there exist contact points R_i of t_i with k_o, there exists half lines $h_i \subset t_i$ of k_o. The distance Δ between t_1 and t_2 is defined as *"wide"* of K_o. If there aren't parallel tangent pairs, then we assume that $\Delta = +\infty$. Under the open H_1, let the endpoints of $\overline{\text{AB}}$ pass through the edge k_o. This is always possible for $\overline{A'B'} < \Delta$. If $\overline{A'B'} = \Delta < \infty$, then the desired motion is possible when the contact points R_i of t_i are exist. For $\overline{A'B'} > \Delta$, the motion is impossible. The points A and B can *turn back* in some cases, during the open motion H_1. The dead centre of an endpoint of $\overline{\text{A}'\text{B}'}$ is a instantaneous rotation pole center at the same time. Also, for each position of the chord $\overline{AB}$ with fixed lenght $a+b$, there exists a parellel tangent of k_o which during the motion makes a complete turn around the total rotation angle δ. Therefore, the total rotation angle $\delta \in \mathbf{R}^+$ of the open H_1 coincides with tangent rotation angle of k_o.

THEOREM 1. Let k_o be the edge of unbounded convex domain $K_o \subset \mathbf{R}^2$ and $\delta \in \mathbf{R}^+$ be its tangent rotation angle. If we assume that the endpoints A and B of the straight line s with length $a+b$ move from the fixed any point along the curve k_0 first to the positive direction and then to the negative direction,the point $X \in s$ ($a = \overline{AX}$, $b = \overline{XB}$) describes a curve $k_o(X)$ and "the Holditch- Sickle" $S_o \subset K_o$, which is bounded by k_o and $k_o(X)$ has the surface area $F_S = h^2(t_0)ab\delta/2$.

PROOF. Let the points $A = (0,0), B = (a+b,0), X = (a,0) \in E$ have the position A^t, B^t, X^t in fixed plane E' for $t > 0$ and analog the position A^{-t}, B^{-t}, X^{-t} for $-t$. Then, these two position can coincide with a rotation round a definite centre $D \in E'$. If the motion $H_1 = E/E'$ is restricted to time interval $[-t,t]$, then an open motion $\tilde{H}_1(t)$ with total rotation angle $\tilde{\delta}(t)$ is obtained. Under $\tilde{H}_1(t)$, the region determined by the center $D \in E'$ and the orbit curve part $\tilde{k}_o(Y)$ of the fixed point $Y = (y_1, y_2) \in E$ has the surface area[1]

$$F_Y^D = F_A^D + \frac{h^2(t_0)\tilde{\delta}(t)}{2}(y_1^2 + y_2^2 - \lambda_1 y_1 - \lambda_2 y_2) + \mu_1 y_1 + \mu_2 y_2, [4] \qquad (3)$$

[1] Using the mean-value theorem for the integral $\int_{-t}^{t} h^2(t)\dot{\varphi}(t)dt$ which occurs during the calculation of the surface area F_Y^D under the motion $\tilde{H}_1(t)$ (we assume that the functions h and φ have the same sign in the interval $[-t,t]$), we get $t_0 \in [-t,t]$ satisfying the equation $h^2(t_0)\int_{-t}^{t} \dot{\varphi}(t)dt = h^2(t_0)\tilde{\delta}(t)$.

for $\exists\, t_0 \in [0,T]$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbf{R}$.

The orbit curve parts $\tilde{k}_o(A)$, $\tilde{k}_o(X)$ of the points A, X and the line segments $A^t X^t$, $A^{-t} X^{-t}$ describe a closed domain. This closed domain has the oriented surface area

$$F_{AX}(t) = F_A^D - F_X^D \tag{4}$$

in order $A^{-t} A^t X^t X^{-t} A^{-t}$. Similarly, we can write

$$F_{AB}(t) = F_A^D - F_B^D. \tag{5}$$

From eqns. (3),(4) and (5), we get

$$F_{AX}(t) = \frac{h^2(t_0) ab\tilde{\delta}(t)}{2} + \frac{a}{a+b} F_{AB}(t). \tag{6}$$

Let $\alpha_A(t)$ and $\alpha_B(t)$ (resp. $\alpha_A(-t)$ and $\alpha_B(-t)$) be the angles between the chord $A^t X^t$ (resp. $A^{-t} X^{-t}$) and k_0. Then, for $F_{AB}(t)$, which is composed of two oriented areas formed by the chord $A^t X^t$ and the curve k_0; and the same at $-t$, we can write for suffiecently large t

$$|F_{AB}(t)| \leq (a+b)^2 [h^2(t)(sin\alpha_A(t) + sin\alpha_B(t)) + h^2(-t)(sin\alpha_A(-t) + sin\alpha_B(-t))].$$

Thus, for $t \to +\infty$, we have

$$\lim_{t\to\infty} F_{AB}(t) = 0. \tag{7}$$

Hence, from eqns. (6) and (7), we get

$$F_S = h^2(t_0) ab\delta/2, \tag{8}$$

where

$$\lim_{t\to\infty} F_{AX}(t) = F_S, \lim_{t\to\infty} \tilde{\delta}(t) = \delta.$$

SPECIAL CASE 1. In the case of the homothetic scale $h \equiv 1$, we have

$$F_S = ab\delta/2$$

which was given by Pottmann [3].

3 Spatial Holditch-Sickles

Let k_A and k_B be normal cross-sections of C^1-cylinder $\Gamma \subset \mathbf{R}^3$ (which is the edge of an unbounded convex domain in $\mathbf{R}^3$) with the planes $z = 0$ and $z = k$. Let the endpoints A, B of a straight line s with constant length l move along congruent convex curves k_A and k_B with tangent rotation angle δ. Then, the straight line s describes a ruled surface whose rulings have the constant angle $\beta = \arcsin(k/hl)$ with the planes $z = 0$ and $z = k$.

The region we defined as spatial Holditch Sickle $\mathrm{S} \subset \mathbf{R}^3$ is the point set bounded by the ruled surface and the cylinder parts between the planes $z = 0$ and $z = k$. During the motion of s, the points $X \in s$ draw planar curves on planes $z = c$, $(0 \leq c \leq k)$. The cross-sections of spatial Holditch-Sickles S in planes $z = c$ is planar Holditch-Sickles with the surface area

$$F_{\mathrm{S}}(c) = h^2(t_0)\frac{\delta}{2}c(k-c)ctg^2\beta. \tag{9}$$

Then, using Cavalieri-Principle, the volume V_{S} of S is

$$V_{\mathrm{S}} = \int_0^k F_{\mathrm{S}}(z)dz. \tag{10}$$

So,using eqns. (9) and (10), we can give the following theorem:

THEOREM 2. The volume of spatial Holditch-Sickles S with the height k, the total rotation angle δ and the slope angle β is

$$V_{\mathrm{S}} = h^2(t_0)\frac{\delta}{2}k^3ctg^2\beta.$$

SPECIAL CASE 2. In the case of the homothetic scale $h \equiv 1$, we have

$$V_{\mathrm{S}} = \frac{\delta}{2}k^3ctg^2\beta,$$

which was given by Pottmann [3].

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