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Orthogonal q-Polynomials Related To Perturbed Linear Form*

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Abstract

The purpose of this paper is to study the regular linear form $\widetilde{u} = \delta_\tau + \lambda(x-\tau)^{-1}u$ where u is H_q-semiclassical. Some q-identities related to this basic class are obtained. An example is carefully analyzed.

1 Introduction

Let u be a regular linear form. We define a new linear form $\widetilde{u}$ by the relation $D(x)\widetilde{u} = A(x)u$ where D and A are non-zero polynomials. This problem has been studied by several authors from different points of view [2,4,7,9,10,12]. In particular, in [12] and for $D(x) = x - \tau$, $\tau \in \mathbb{C}$, $A(x) = \lambda$, $\lambda \in \mathbb{C} - \{0\}$, P. Maroni found necessary and sufficient conditions to $\widetilde{u}$ to be regular. So, the aim of our contribution is to study the H_q-semiclassical character of $\widetilde{u}$ by taking into account theory of H_q-semiclassical orthogonal polynomials in [5,6] which is a basic class of the so-called discrete orthogonal polynomials with H_q the q-derivative operator[3,5,6,8]. In particular, the class $\widetilde{s}$ of $\widetilde{u}$ is discussed. Also the structure relation and the second order linear q-difference equation of the (MOPS) associated with $\widetilde{u}$ are established. Finally, the perturbation of the little q-Laguerre H_q-classical linear form is treated.

2 Preliminaries and Notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. The linear form u is called regular if we can associate with it a polynomial sequence $\{P_n\}_{n\geq 0}$, $\deg P_n = n$, such that $\langle u, P_mP_n\rangle = k_n\delta_{n,m}$, $n, m \geq 0$, $k_n \neq 0$, $n \geq 0$; the left multiplication gu is defined by $\langle gu, f\rangle := \langle u, gf\rangle$. Similarly, we define $\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle$, $u \in \mathcal{P}'$, $f \in \mathcal{P}$, $a \in \mathbb{C} - \{0\}$. We consider the following well known problem: given a regular linear form u, find all regular linear form $\widetilde{u}$ which satisfy the following equation

$$(x-\tau)\widetilde{u} = \lambda u \ , \ \tau \in \mathbb{C} \ , \ \lambda \in \mathbb{C} - \{0\}, \tag{1}$$

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with constraints $(\widetilde{u})_0 = 1$, $(u)_0 = 1$, where $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, are the moments of u. Equivalently, $\widetilde{u} = \delta_\tau + \lambda(x-\tau)^{-1}u$ where $\langle \delta_\tau, f \rangle = f(\tau)$ and the linear form $(x-\tau)^{-1}u$ is defined by $\langle (x-\tau)^{-1}u, f \rangle := \langle u, \theta_\tau f \rangle$, with in general $(\theta_\tau f)(x) := \dfrac{f(x)-f(\tau)}{x-\tau}$. In particular, $\lambda + \tau = (\widetilde{u})_1$. If we suppose that the linear form u possesses the discrete representation

$$u = \sum_{n \geq 0} \rho_n \delta_{\tau_n}, \tag{2}$$

where $\Big| \sum_{n \geq 0} \rho_n (\tau_n)^p \Big| < +\infty$, $p \geq 0$, then the linear form $\widetilde{u}$ is represented by

$$\widetilde{u} = \Big\{ 1 - \lambda \sum_{n \geq 0} \frac{\rho_n}{\tau_n - \tau} \Big\} \delta_\tau + \lambda \sum_{n \geq 0} \frac{\rho_n}{\tau_n - \tau} \delta_{\tau_n}, \tag{3}$$

since

$$\Big| \sum_{n \geq 0} \frac{\rho_n}{\tau_n - \tau} (\tau_n)^p \Big| < +\infty, \; p \geq 0. \tag{4}$$

In accordance with (1) and after some calculations, we are able to give the connection between the moments of $\widetilde{u}$ and u

$$(\widetilde{u})_n = \tau^n + \lambda \sum_{\nu=1}^{n} \tau^{n-\nu} (u)_{\nu-1}, \; n \geq 1. \tag{5}$$

Let $\{P_n\}_{n \geq 0}$ denote the sequence of orthogonal polynomials with respect to u

$$P_0(x) = 1 \; , \; P_1(x) = x - \beta_0 \; , \; P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x) \; , \; n \geq 0. \tag{6}$$

Suppose $\widetilde{u}$ is regular and let $\{\widetilde{P}_n\}_{n \geq 0}$ be its corresponding orthogonal sequence

$$\widetilde{P}_0(x) = 1 \; , \; \widetilde{P}_1(x) = x - \widetilde{\beta}_0 \; , \; \widetilde{P}_{n+2}(x) = (x - \widetilde{\beta}_{n+1}) \widetilde{P}_{n+1}(x) - \widetilde{\gamma}_{n+1} \widetilde{P}_n(x) \; , \; n \geq 0. \tag{7}$$

The relationship between $\widetilde{P}_n$ and P_n is (see [12])

$$\widetilde{P}_{n+1}(x) = P_{n+1}(x) + a_n P_n(x) \; , \; a_n = -\frac{P_{n+1}(\tau) + \lambda P_n^{(1)}(\tau)}{P_n(\tau) + \lambda P_{n-1}^{(1)}(\tau)} \neq 0 \; , \; n \geq 0, \tag{8}$$

where $P_n^{(1)}(x) := \langle u, \dfrac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi} \rangle$, $n \geq 0$. We have [11]

$$P_{n+1}^{(1)}(x) P_{n+1}(x) - P_{n+2}(x) P_n^{(1)}(x) = \prod_{k=0}^{n} \gamma_{k+1} \; , \; n \geq 0. \tag{9}$$

Set

$$\lambda_n = -\frac{P_n(\tau)}{P_{n-1}^{(1)}(\tau)} \; , \; n \geq 1 \; , \; \lambda_0 = 0. \tag{10}$$

Let us recall that the linear form $\widetilde{u} = \delta_\tau + \lambda(x-\tau)^{-1}u$ is regular if and only if $\lambda \neq \lambda_n,\ n \geq 0$. In this case we may write [12]

$$\frac{\gamma_{n+1}}{a_n} + a_{n+1} - \beta_{n+1} = -\tau\ ,\ n \geq 0, \tag{11}$$

$$\widetilde{\beta}_0 = \beta_0 - a_0 = \tau + \lambda\ ,\ \widetilde{\beta}_{n+1} = \beta_{n+1} + a_n - a_{n+1}\ ,\ \widetilde{\gamma}_{n+1} = -a_n(a_n - \beta_n + \tau)\ ,\ n \geq 0, \tag{12}$$

$$\begin{cases} (x-\tau)P_n(x) = \widetilde{P}_{n+1}(x) + (\beta_n - a_n - \tau)\widetilde{P}_n(x)\ ,\ n \geq 0\ , \\ \\ (x-\tau)P_{n+1}(x) = (x - a_n - \tau)\widetilde{P}_{n+1}(x) + a_n(a_n - \beta_n + \tau)\widetilde{P}_n(x)\ ,\ n \geq 0. \end{cases} \tag{13}$$

Let us introduce the q-derivative operator H_q by $(H_q f)(x) = \dfrac{f(qx) - f(x)}{qx - x},\ f \in \mathcal{P}$. By duality, we can define H_q from $\mathcal{P}'$ to $\mathcal{P}'$ such that $\langle H_q u, f\rangle = -\langle u, H_q f\rangle,\ f \in \mathcal{P}$, $u \in \mathcal{P}'$. In particular, this yields $(H_q u)_n = -[n]_q (u)_{n-1},\ n \geq 0$ with $(u)_{-1} = 0$ and $[n]_q := \dfrac{q^n - 1}{q - 1}, n \geq 0$. [3,5,6,8]

The linear form u is said to be H_q-semiclassical when it is regular and there exists two polynomials Φ (monic) and Ψ with $\deg \Phi \geq 0$, $\deg \Psi \geq 1$ such that

$$H_q(\Phi u) + \Psi u = 0. \tag{14}$$

The class of the H_q-semiclassical linear form u is $s = \max(\deg \Phi - 2, \deg \Psi - 1) \geq 0$ if and only if the following condition is satisfied

$$\prod_{c \in \mathcal{Z}_\Phi} \Big\{ |q\,(h_q \Psi)\,(c) + (H_q \Phi)\,(c)| + |\langle u, q\,(\theta_{cq}\Psi) + (\theta_{cq} \circ \theta_c \Phi)\rangle| \Big\} > 0, \tag{15}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ [6]. We can state characterizations of the corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ as follows: [6]

1). $\{P_n\}_{n\geq 0}$ satisfies the following structure relation

$$\Phi(x)(H_q P_{n+1})(x) = \frac{C_{n+1}(x) - C_0(x)}{2} P_{n+1}(x) - \gamma_{n+1} D_{n+1}(x) P_n(x),\ n \geq 0, \tag{16}$$

where

$$\begin{cases} C_{n+1}(x) = -C_n(x) + 2(x - \beta_n) D_n(x) + 2x(q-1)\Sigma_n(x)\ ,\ n \geq 0\ , \\ \gamma_{n+1} D_{n+1}(x) = -\Phi(x) + \gamma_n D_{n-1}(x) + (x - \beta_n)^2 D_n(x) - \\ -(\frac{q+1}{2}x - \beta_n) C_n(x) + x(q-1)\{\frac{1}{2} C_0(x) + (x - \beta_n)\Sigma_n(x)\},\ n \geq 0\ , \\ \Sigma_n(x) := \displaystyle\sum_{k=0}^{n} D_k(x)\ ,\ n \geq 0\ , C_0(x) = -\big(q(h_q \Psi)(x) + (H_q \Phi)(x)\big)\ , \\ D_0(x) = -\big(H_q(u\theta_0 \Phi)(x) + q h_q(u\theta_0 \Psi)(x)\big)\ ,\ D_{-1}(x) := 0, \end{cases} \tag{17}$$

with $(uf)(x) := \langle u, \dfrac{xf(x) - \xi f(\xi)}{x - \xi}\rangle\ ,\ f \in \mathcal{P}$. Φ, Ψ are the same polynomials as in (14); β_n, γ_n are the coefficients of the three term recurrence relation (6). Notice that $\deg C_n \leq s + 1\ , \deg D_n \leq s\ , n \geq 0$.

2). Also, each polynomial P_{n+1}, $n \geq 0$, satisfies a second order linear $q-$difference equation. For $n \geq 0$

$$J_q(x,n)(H_q \circ H_{q^{-1}} P_{n+1})(x) + K_q(x,n)(H_{q^{-1}} P_{n+1})(x) + L_q(x,n) P_{n+1}(x) = 0, \quad (18)$$

with

$$\begin{cases} J_q(x,n) = q\Phi(x) D_{n+1}(x) \ , \\ K_q(x,n) = D_{n+1}(q^{-1}x)(H_{q^{-1}}\Phi)(x) - (H_{q^{-1}} D_{n+1})(x)\Phi(q^{-1}x) + \\ \qquad\qquad\qquad + C_0(q^{-1}x) D_{n+1}(x) \ , \\ L_q(x,n) = \frac{1}{2}(C_{n+1}(q^{-1}x) - C_0(q^{-1}x))(H_{q^{-1}} D_{n+1})(x) - \\ -\frac{1}{2}(H_{q^{-1}}(C_{n+1} - C_0))(x) D_{n+1}(q^{-1}x) - D_{n+1}(x)\Sigma_n(q^{-1}x) \ , \ n \geq 0. \end{cases} \quad (19)$$

Φ, C_n, D_n are the same as in the previous characterization. Notice that $\deg J_q(.,n) \leq 2s+2$, $\deg K_q(.,n) \leq 2s+1$, $\deg L_q(.,n) \leq 2s$. In particular, when $s = 0$ that is to say the H_q-classical case, the coefficients of the structure relation (16) become [6]

$$\begin{cases} \dfrac{C_{n+1}(x) - C_0(x)}{2} = \dfrac{1}{2}\Phi''(0)([n+1]_q x - q^{-n-1}S_n) + \\ + q^{-n-1}(\Psi'(0) - \frac{1+q^{n+1}}{2}\Phi''(0)[n+1]_q)\beta_{n+1} + \\ \quad + q^{-n-1}(\Psi(0) - \Phi'(0)[n+1]_q) - q^{-n-1}(q-1)\Psi'(0) S_n \\ D_{n+1}(x) = q^{-n}\left\{\frac{1}{2}\Phi''(0)[2n+1]_q - \Psi'(0)\right\}, \ n \geq 0, \end{cases} \quad (20)$$

with $S_n = \sum_{k=0}^{n} \beta_k, n \geq 0$. Also we get for(19) [6]

$$\begin{cases} J_q(x,n) = \Phi(x) \ , \\ K_q(x,n) = -\Psi(x) \ , \\ L_q(x,n) = q^{-n}[n+1]_q(\Psi'(0) - \frac{1}{2}\Phi''(0)[n]_q) \ , \ n \geq 0. \end{cases} \quad (21)$$

3 The H_q-Semiclassical Case

3.1 The H_q-semiclassical character of $\widetilde{u}$

In the sequel the linear form u will be supposed to be H_q-semiclassical of class s satisfying the q-Pearson equation $H_q(\Phi u) + \Psi u = 0$. From (1), it is clear that the linear form $\widetilde{u}$, when it is regular, is also H_q-semiclassical and satisfies

$$H_q(\widetilde{\Phi}\widetilde{u}) + \widetilde{\Psi}\widetilde{u} = 0, \quad (22)$$

with

$$\widetilde{\Phi}(x) = (x-\tau)\Phi(x) \text{ and } \widetilde{\Psi}(x) = (x-\tau)\Psi(x). \quad (23)$$

The class of $\widetilde{u}$ is at most $\widetilde{s} = s+1$.

PROPOSITION 1. The class of $\widetilde{u}$ depends only on the zero $x = \tau q^{-1}$.

For the proof we use the following lemma:

LEMMA 1. For all root c of Φ we have

$$\langle \widetilde{u}, q\theta_{cq}\widetilde{\Psi} + (\theta_{cq} \circ \theta_c\widetilde{\Phi})\rangle = q(h_q\Psi)(c) + (H_q\Phi)(c) + \lambda\langle u, q\theta_{cq}\Psi + (\theta_{cq} \circ \theta_c\Phi)\rangle \tag{24}$$

and

$$q(h_q\widetilde{\Psi})(c) + (H_q\widetilde{\Phi})(c) = (cq - \tau)\big\{q(h_q\Psi)(c) + (H_q\Phi)(c)\big\}. \tag{25}$$

PROOF. Let c be a root of Φ, then we can write

$$\widetilde{\Phi}(x) = (x - \tau)(x - c)\Phi_c(x) \text{ and } \Phi_c(x) = (\theta_c\Phi)(x). \tag{26}$$

So from (23) and (26) we have

$$\langle \widetilde{u}, q\theta_{cq}\widetilde{\Psi} + (\theta_{cq} \circ \theta_c\widetilde{\Phi})\rangle = q\langle \widetilde{u}, \theta_{cq}\big((\xi - \tau)\Psi\big)\rangle + \langle \widetilde{u}, \theta_{cq}\big((\xi - \tau)\Phi_c\big)\rangle. \tag{27}$$

Using the definition of the operator θ_c, it is easy to prove that

$$\theta_c(fg)(x) = g(x)(\theta_c f)(x) + f(c)(\theta_c g)(x),\ \forall f, g \in \mathcal{P}. \tag{28}$$

Taking $g(x) = x - \tau$ and $f(x) = \Phi_c(x)$, we obtain

$$\begin{aligned} \langle \widetilde{u}, \theta_{cq}\big((\xi - \tau)\Phi_c\big)\rangle &= \langle \widetilde{u}, (x - \tau)\big(\theta_{cq}\Phi_c\big)(x) + \Phi_c(cq)\rangle \\ &= \langle \widetilde{u}, (x - \tau)\big(\theta_{cq} \circ \theta_c\Phi\big)(x)\rangle + (H_q\Phi)(c) \end{aligned}$$

because

$$\theta_{cq}\Phi_c = \theta_{cq} \circ \theta_c\Phi,\ \Phi_c(cq) = (H_q\Phi)(c) \text{ and } (\theta_{cq}(\xi - \tau))(x) = 1.$$

By virtue of (1) we get

$$\langle \widetilde{u}, \theta_{cq}\big((\xi - \tau)\Phi_c\big)\rangle = \lambda\langle u, \theta_{cq} \circ \theta_c\Phi\rangle + (H_q\Phi)(c). \tag{29}$$

Now, taking $g(x) = x - \tau$ and $f(x) = \Psi(x)$ in (28), we obtain

$$q\langle \widetilde{u}, \theta_{cq}\big((\xi - \tau)\Psi\big)\rangle = q\langle \widetilde{u}, (x - \tau)(\theta_{cq}\Psi)(x) + \Psi(cq)\rangle.$$

Taking (1) into account we get

$$q\langle \widetilde{u}, \theta_{cq}\big((\xi - \tau)\Psi\big)\rangle = q\lambda\langle u, \theta_{cq}\Psi\rangle + (h_q\Psi)(c). \tag{30}$$

Replacing (29) and (30) in (27), we obtain (24). Also (25) is deduced.

PROOF OF PROPOSITION 1. Let c be a root of Φ such that $c \neq \tau q^{-1}$.
If $q(h_q\Psi)(c) + (H_q\Phi)(c) = 0$, from (24) we have $\langle \widetilde{u}, q\theta_{cq}\widetilde{\Psi} + (\theta_{cq} \circ \theta_c\widetilde{\Phi})\rangle \neq 0$ since u is H_q-semiclassical of class s and so satisfies (15).
If $q(h_q\Psi)(c) + (H_q\Phi)(c) \neq 0$, then $q(h_q\widetilde{\Psi})(c) + (H_q\widetilde{\Phi})(c) \neq 0$ from (25).
In any case, we cannot simplify by $(x - c)$.

As a consequence we get the following result:

COROLLARY 1. If the H_q-semiclassical linear form u is of class s then the linear form $\widetilde{u}$ is H_q-semiclassical of class $\widetilde{s} = s + 1$ for

$$\Phi(\tau q^{-1}) \neq 0 \ , \ \lambda \neq \lambda_n, \ n \geq 0 \ \text{ or } \ \Phi(\tau q^{-1}) = 0 \ , \ \lambda \neq \lambda_n, \ n \geq -1, \tag{31}$$

where

$$\lambda_{-1} = -\frac{q\Psi(\tau) + (H_{q^{-1}}\Phi)(\tau)}{\langle u, q\theta_\tau \Psi + \theta_\tau \circ \theta_{\tau q^{-1}} \Phi \rangle}. \tag{32}$$

3.2 The structure relation and the second order linear q-difference equation of $\{\widetilde{P}_n\}_{n\geq 0}$

From (8), (16) and (6) we have for $n \geq 0$

$$\Phi(x)(H_q \widetilde{P}_{n+1})(x) = u_n(x) P_{n+1}(x) + v_n(x) P_n(x), \tag{33}$$

$$\begin{cases} u_n(x) = \frac{1}{2}(C_{n+1}(x) - C_0(x)) + a_n D_n(x), \\ v_n(x) = \Big(-\frac{1}{2}(C_{n+1}(x) - C_0(x)) - C_0(x) + \\ \qquad\qquad +x(q-1)\Sigma_n(x) \Big) a_n - \gamma_{n+1} D_{n+1}(x). \end{cases} \tag{34}$$

On account of (13) and the fact that $P_{n+1}(x)$ and $P_n(x)$ are coprime, we have for (33) for $n \geq 0$

$$\widetilde{\Phi}(x)(H_q \widetilde{P}_{n+1})(x) = \frac{1}{2}(\widetilde{C}_{n+1}(x) - \widetilde{C}_0(x))\widetilde{P}_{n+1}(x) - \widetilde{\gamma}_{n+1} \widetilde{D}_{n+1}(x) \widetilde{P}_n(x), \tag{35}$$

where

$$\begin{cases} \frac{1}{2}(\widetilde{C}_{n+1}(x) - \widetilde{C}_0(x)) = (x - \tau - a_n) u_n(x) + v_n(x) \\ \widetilde{\gamma}_{n+1} \widetilde{D}_{n+1}(x) = (a_n - \beta_n + \tau)(v_n(x) - a_n u_n(x)) \end{cases} \ , \ n \geq s+1. \tag{36}$$

From (17) we have

$$\widetilde{C}_0(x) = -q(h_q \widetilde{\Psi})(x) - (H_q \widetilde{\Phi})(x) \ , \ \widetilde{D}_0(x) = -H_q(\widetilde{u}\theta_0 \widetilde{\Phi})(x) - q h_q(\widetilde{u}\theta_0 \widetilde{\Psi})(x).$$

By virtue of (23) we get

$$\widetilde{C}_0(x) = (qx - \tau) C_0(x) - \Phi(x) \ , \ \widetilde{D}_0(x) = C_0(x) + \lambda D_0(x), \tag{37}$$

because

$$\begin{aligned} (\widetilde{u}\theta_0 \widetilde{\Psi})(x) &= \langle \widetilde{u}, \frac{\widetilde{\Psi}(x) - \widetilde{\Psi}(\xi)}{x - \xi} \rangle \\ &= \Psi(x) + \langle \lambda(\xi - \tau)^{-1} u, \frac{\widetilde{\Psi}(x) - \widetilde{\Psi}(\xi)}{x - \xi} \rangle \\ &= \Psi(x) + \lambda \langle u, \Big\{ \frac{\widetilde{\Psi}(x) - \widetilde{\Psi}(\xi)}{x - \xi} - \Psi(x) \Big\} \frac{1}{\xi - \tau} \rangle \\ &= \Psi(x) + \lambda (u\theta_0 \Psi)(x). \end{aligned}$$

Consequently and by virtue of (17), we can easily prove by induction that the system (36) is valid for $0 \leq n \leq s$. Hence (36) is valid for $n \geq 0$.
In addition, from (34)-(37) and by taking into account (11) and (17) we get for $n \geq 0$

$$\widetilde{\Sigma}_n(x) := \sum_{\nu=0}^{n} \widetilde{D}_\nu(x) = -\frac{1}{2}(C_{n+1}(x) - C_0(x)) - a_n D_n(x) + (qx - \tau)\Sigma_n(x). \qquad (38)$$

Therefore, the coefficients of the second order linear q-difference equation satisfied by $\widetilde{P}_{n+1},\ n \geq 0$ are for $n \geq 0$

$$\begin{cases} \widetilde{J}_q(x,n) = q(x-\tau)\Phi(x)\big(v_n(q^{-1}x) - a_n u_n(q^{-1}x)\big), \\ \widetilde{K}_q(x,n) = \Big\{\big(v_n(q^{-1}x) - a_n u_n(q^{-1}x)\big)\times \\ \quad \big(\Phi(x) + (q^{-1}x - \tau)(H_{q^{-1}}\Phi)(x)\big)\Big\} - \\ \qquad -\Big\{\big(v_n(x) - a_n u_n(x)\big)\times \\ \Big((x-\tau)\big(q\Psi(x) + (H_{q^{-1}}\Phi)(x)\big) + \Phi(q^{-1}x)\Big)\Big\} - \\ \quad -\big((H_{q^{-1}}v_n)(x) - a_n(H_{q^{-1}}u_n)(x)\big)(q^{-1}x - \tau)\Phi(q^{-1}x), \\ \widetilde{L}_q(x,n) = -\Big\{\big(v_n(q^{-1}x) - a_n u_n(q^{-1}x)\big)\times \\ \quad \big(u_n(x) + (q^{-1}x - \tau - a_n)(H_{q^{-1}}u_n)(x)\big)\Big\} + \\ \qquad +\Big\{\big((H_{q^{-1}}v_n)(x) - a_n(H_{q^{-1}}u_n)(x)\big)\times \\ \qquad \big((q^{-1}x - \tau - a_n)u_n(q^{-1}x) + v_n(q^{-1}x)\big)\Big\} + \\ \qquad\quad +\big(v_n(x) - a_n u_n(x)\big)\big(u_n(x) - \Sigma_n(x)\big). \end{cases} \qquad (39)$$

3.3 An Illustrative Example

First, let us recall the following standard material needed to the sequel[1,5,6]

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{\nu=1}^{n}(1 - aq^{\nu-1}),\ n \geq 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},\ 0 \leq k \leq n\ ,$$

and

$$(a;q)_\infty = \prod_{\nu=0}^{+\infty}(1 - aq^\nu)\ ,\ |q| < 1; \sum_{\nu=0}^{+\infty} \frac{(a;q)_\nu}{(q;q)_\nu} z^\nu = \frac{(az;q)_\infty}{(z;q)_\infty}\ ,\ |z| < 1, |q| < 1.$$

Second, let us consider the H_q-classical linear form $u = u(a,q)$ of little q-Laguerre for $0 < q < 1$ and $0 < a < q^{-1}$. From (17), (20) and (21), and by virtue of [5] we get

Table 1.

β_n	$\{1+a-a(1+q)q^n\}q^n,\ n\geq 0.$	
γ_{n+1}	$a(1-q^{n+1})(1-aq^{n+1})q^{2n+1},\ n\geq 0.$	
$\Phi(x)$	x	
$\Psi(x)$	$-(aq)^{-1}(q-1)^{-1}\{x-1+aq\}.$	
u	$(aq;q)_\infty \sum_{\nu=0}^{+\infty} \frac{(aq)^\nu}{(q;q)_\nu}\delta_{q^\nu}, 0<q<1, 0<a<q^{-1}.$	
$(u)_n$	$(aq;q)_n, n\geq 0.$	
$\frac{C_{n+1}(x)-C_0(x)}{2}$	$[n+1]_q,\ n\geq 0.$	
$D_{n+1}(x)$	$(aq)^{-1}(q-1)^{-1}q^{-n},\ n\geq 0.$	
$C_0(x)$	$a^{-1}(q-1)^{-1}\{qx+a-1\}.$	
$D_0(x)$	$a^{-1}(q-1)^{-1}.$	
$J_q(x,n)$	$x,\ n\geq 0.$	
$K_q(x,n)$	$(aq)^{-1}(q-1)^{-1}\{x-1+aq\},\ n\geq 0.$	
$L_q(x,n)$	$-(aq)^{-1}(q-1)^{-1}q^{-n}[n+1]_q,\ n\geq 0.$	

Putting $x=0$ in (16) and with Table 1, we get $P_{n+1}(0)=-q^n(1-aq^{n+1})P_n(0), n\geq 0$. Consequently,

$$P_n(0)=(-1)^n q^{\frac{(n-1)n}{2}}(aq;q)_n,\ n\geq 0. \tag{40}$$

Moreover, taking $x=0$ in (9), in accordance of Table 1 and (40), an easy computation leads to

$$P_n^{(1)}(0)=(-1)^n q^{\frac{(n+1)n}{2}}(aq;q)_{n+1}\sum_{k=0}^{n}\frac{(q;q)_k}{(aq;q)_{k+1}}a^k\neq 0 \tag{41}$$

for $n\geq 0$, $0<q<1$ and $0<a<q^{-1}$.
Thus, we obtain for (8) and (10)

$$a_n=q^n(1-aq^{n+1})\frac{1-\lambda\xi_{n+1}}{1-\lambda\xi_n},\ n\geq 0, \tag{42}$$

$$\lambda_n=\xi_n^{-1},\ n\geq 1, \lambda_0=0, \tag{43}$$

where

$$\xi_n = \sum_{k=0}^{n-1} \frac{(q;q)_k}{(aq;q)_{k+1}} a^k,\ n \geq 1,\ \ \xi_0 = 0.$$

Consequently, on account of Corollary 1 and (23), (31), (32), the linear form $\widetilde{u} = \delta_0 + \lambda x^{-1} u$ is H_q-semiclassical of class $\widetilde{s} = 1$ for any $\lambda \neq \lambda_n, n \geq -1$ with $\lambda_{-1} = 1 - a$ and fulfils the functional equation (22) with

$$\widetilde{\Phi}(x) = x^2\ ,\ \widetilde{\Psi}(x) = -(aq)^{-1}(q-1)^{-1} x\{x - 1 + aq\}. \tag{44}$$

From (5) with $\tau = 0$ and Table 1, the moments of $\widetilde{u}$ are

$$(\widetilde{u})_0 = 1\ ,\ (\widetilde{u})_n = \lambda (aq;q)_{n-1}, n \geq 1. \tag{45}$$

In addition, regarding (3) the linear form $\widetilde{u}$ is represented by the following discrete measure

$$\widetilde{u} = (aq;q)_\infty \Big\{ (1 - \frac{\lambda}{(a;q)_\infty})\delta_0 + \lambda \sum_{n=0}^{+\infty} \frac{a^n}{(q;q)_n} \delta_{q^n} \Big\}\ ,\ 0 < a < 1, 0 < q < 1. \tag{46}$$

Indeed, (4) is fulfilled, for, putting $w_n(p) = \dfrac{a^n}{(q;q)_n} q^{np}, n, p \geq 0$, we have

$$\frac{w_{n+1}(p)}{w_n(p)} = \frac{aq^p}{1 - q^{n+1}} \longrightarrow aq^p, n \to +\infty, \forall p \geq 0$$

and $aq^p < 1, \forall p \geq 0$ if and only if $a < 1$.
Also, by virtue of (11)-(12) and Table 1, we obtain successively

$$\widetilde{\beta}_0 = \lambda\ ;\ \widetilde{\beta}_{n+1} = q^n \Big\{ aq(1 - q^{n+1}) \frac{1 - \lambda\xi_n}{1 - \lambda\xi_{n+1}} + (1 - aq^{n+1}) \frac{1 - \lambda\xi_{n+1}}{1 - \lambda\xi_n} \Big\},\ n \geq 0, \tag{47}$$

$$\widetilde{\gamma}_1 = \lambda(1 - aq - \lambda); \widetilde{\gamma}_{n+1} = aq^{2n}(1 - q^n)(1 - aq^{n+1}) \frac{(1 - \lambda\xi_{n-1})(1 - \lambda\xi_{n+1})}{(1 - \lambda\xi_n)^2}, n \geq 1. \tag{48}$$

Finally, we have all components to write the structure relation and the second order linear q-difference equation of $\widetilde{P}_n$ according to (34)-(39).

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