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Unitary Completions Of Complex Symmetric And Skew Symmetric Matrices*

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Abstract

Unitary symmetric completions of complex symmetric matrices are obtained via Autonne decomposition. The problem arises from atomic physics. Of independent interest unitary skew symmetric completions of skew symmetric matrices are also obtained by Hua decomposition.

1 Introduction

A recent theory in atomic physics, called phase-integral halfway-house variational continuum distorted wave theory (PIVCDW) [1], requires finding a symmetric unitary matrix X (that is, X is coninvolutory: $X\overline{X} = I$) whose leading principal submatrix is $S/\|S\|$, where $S \in \mathbb{C}_{n\times n}$ is a given complex nonsingular symmetric matrix. The resulting matrix X can be applied to correct a loss of unitarity of a scattering matrix due to the use of a finite basis set in the solution of a collision problem. We call the problem a *unitary symmetric completion* of the symmetric matrix S. Brown and Crothers [1] studied the problem and obtained the following result for providing some unitary completions by rather lengthy computation. Let S^* denote the complex conjugate transpose of S.

THEOREM 1.1 (Brown and Crothers). Let S be a complex symmetric non-singular matrix with singular values $s_1 \geq s_2 \geq \cdots \geq s_n$. Let w_j, $j = 1, \dots, n$, be the (unit) eigenvectors of SS^* corresponding to the eigenvalues of s_j^2, $j = 1, \dots, n$. The complex symmetric matrix

$$X = \frac{1}{\|S\|}\begin{pmatrix} S & A \\ A^T & Z \end{pmatrix} \tag{1}$$

is unitary, where

$$A = [w_2 \cdots w_n]\,\mathrm{diag}\,((s_1^2 - s_2^2)^{1/2}e^{i\theta_2}, \dots, (s_1^2 - s_n^2)^{1/2}e^{i\theta_n}) \in \mathbb{C}_{n\times(n-1)},$$

$$Z = \mathrm{diag}\,(s_2 e^{i\phi_2}, \dots, s_n e^{i\phi_n}),$$

and

$$\phi_j = \pi - \arg\,(w_j^T S \overline{w}_j) - 2\theta_j, \quad j = 2, \dots, n. \tag{2}$$

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The matrix X in (1) is smallest in size.

However Theorem 1.1 is incorrect, but can be easily fixed, due to a minor error in [1, p.2927] (which occurred when formula (23) was deduced from (21) incorrectly).

EXAMPLE 1.2. Let $S = \operatorname{diag}(s_1, s_2)$ with $s_1 > s_2 > 0$. According to the construction in Theorem 1.1, pick $w_1 = e_1, w_2 = e_2$, where $I_2 = [e_1\, e_2]$, and

$$Z = (s_2 e^{i\phi_2}), \qquad A = \begin{pmatrix} 0 \\ (s_1 - s_2)^{1/2} e^{i\theta_2} \end{pmatrix},$$

with $\theta_2, \phi_2 \in \mathbb{R}$ so that

$$X = \frac{1}{s_1} \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & (s_1^2 - s_2^2)^{1/2} e^{i\theta_2} \\ 0 & (s_1^2 - s_2^2)^{1/2} e^{i\theta_2} & s_2 e^{i\phi_2} \end{pmatrix}.$$

Notice that $\arg(w_2^T S \overline{w}_2) = 0$. For X to be a unitary matrix, we must have $e^{2i\theta_2} + e^{i\phi_2} = 0$. Now $\phi_2 = \pi + 2\theta_2$ would work but $\phi_2 = \pi - 2\theta_2$ ($\theta_2 \neq 0$) would not in general.

In [1, p.2927-2928] a unitary completion was given for an $S \in \mathbb{C}_{3\times 3}$ but the choice $\theta_2 = \theta_3 = 0$ was made so that the error (2) was not manifested. The condition (2) should be replaced by

$$\phi_j = \pi - \arg(w_j^T S \overline{w}_j) + 2\theta_j, \quad j = 2, \ldots, n. \tag{3}$$

With the above adjustment, Theorem 1.1 still fails to be true if some eigenvalue of SS^* is not simple.

EXAMPLE 1.3. Let $S = \operatorname{diag}(\sqrt{3}, 1, 1)$. The eigenvalues of SS^* are 3, 1, 1. Clearly $w_2 = (1/\sqrt{2})\,(0, 1, i)^T$ and $w_3 = (1/\sqrt{2})\,(0, 1, -i)^T$ are eigenvectors of SS^* corresponding to 1. According to the construction in Theorem 1.1,

$$\begin{aligned} A &= \sqrt{2}[e^{i\theta_2} w_2 \,|\, e^{i\theta_3} w_3], \\ Z &= \operatorname{diag}(e^{i\phi_2}, e^{i\phi_3}), \\ XX^* &= \frac{1}{3} \begin{pmatrix} SS^* + AA^* & S\overline{A} + AZ^* \\ A^T S^* + ZA^* & ZZ^* + A^T \overline{A} \end{pmatrix}. \end{aligned}$$

However

$$\begin{aligned} S\overline{A} + AZ^* &= [e^{-i\theta_2} \overline{w}_2 + e^{i(\theta_2 - \phi_2)} w_2 \,|\, e^{-i\theta_3} \overline{w}_3 + e^{i(\theta_3 - \phi_3)} w_3] \\ &= [e^{-i\theta_2} w_3 + e^{i(\theta_2 - \phi_2)} w_2 \,|\, e^{-i\theta_3} w_2 + e^{i(\theta_3 - \phi_3)} w_3] \\ &\neq 0_{3\times 2}, \end{aligned}$$

for any θ_j and ϕ_j, $j = 2, 3$, since w_2 and w_3 are linearly independent. So X is not unitary.

Our first goal is to give a complete description of all possible unitary symmetric completions of a general complex symmetric matrix $S \in \mathbb{C}_{n\times n}$. The result, Theorem 2.1, is given in Section 2 by using Autonne decomposition. The advantage of using Autonne decomposition is that it provides canonical unit eigenvectors for SS^* without getting into the trouble in Example 1.3. Based on Theorem 2.1, we remark that (1) If s_1 is not simple, the size of X could be even smaller. (2) The singular case is also handled. (3) Theorem 1.1 does not sort out all possible unitary completions, even those of smallest size.

Of independent interest, we study the unitary completion problem for skew symmetric matrices in Section 3.

2 Unitary Symmetric Completions for Symmetric Matrices

The singular values of a matrix $A \in \mathbb{C}_{n\times n}$ are the square roots of the eigenvalues of AA^* or A^*A. We will find all unitary symmetric completions via Autonne decomposition of a complex symmetric matrix $S \in \mathbb{C}_{n\times n}$ [4, p.204-205] which asserts that there is a unitary $U \in \mathbb{C}_{n\times n}$ such that

$$U^T SU = s_1 I_{n_1} \oplus s_2 I_{n_2} \oplus \cdots \oplus s_k I_{n_k}, \tag{4}$$

where $s_1 > s_2 > \cdots > s_k$ are the distinct singular values of S and s_j has multiplicity n_j, $j = 1, \ldots, k$ $(n_1 + \cdots + n_k = n)$.

THEOREM 2.1. Let $S \in \mathbb{C}_{n\times n}$ be a nonzero complex symmetric matrix with Autonne decomposition (4). Then

$$X = \frac{1}{\|S\|}\begin{pmatrix} S & A \\ A^T & Z \end{pmatrix} \in \mathbb{C}_{(n+m)\times(n+m)}, \tag{5}$$

is a unitary symmetric matrix if and only if

1. the distinct singular values of Z are $s_1 > s_2 > \cdots > s_k$ where s_j has multiplicity n_j, $j = 2, \ldots, k$, and s_1 has multiplicity $m-n+n_1$, that is, there exists a unitary matrix $V \in \mathbb{C}_{m\times m}$ such that

   $$V^T ZV = s_1 I_{m-n+n_1} \oplus s_2 I_{n_2} \oplus \cdots \oplus s_k I_{n_k} \in \mathbb{C}_{m\times m},$$

   and

2. $$U^T AV = \begin{pmatrix} 0_{n_1\times(m-n+n_1)} & 0 \\ 0 & A_0 \end{pmatrix},$$

   where $A_0 \in \mathbb{C}_{(n-n_1)\times(n-n_1)}$ is of the form

   $$A_0 = (s_1^2 - s_2^2)^{1/2} A_2 \oplus (s_1^2 - s_3^2)^{1/2} A_3 \oplus \cdots \oplus (s_1^2 - s_k^2)^{1/2} A_k,$$

   and $iA_j \in \mathbb{C}_{n_j\times n_j}$, $j = 2, \ldots, k$, are orthogonal matrices except for iA_k when $s_k = 0$ (in which case A_k is unitary).

When $m = n - n_1$, the unitary completions (5) are smallest in size.

PROOF. Let
$$X = \frac{1}{s_1}\begin{pmatrix} S & A \\ A^T & Z \end{pmatrix} \in \mathbb{C}_{(n+m)\times(n+m)}$$
be a unitary symmetric completion of $S \in \mathbb{C}_{n\times n}$. Let $Z_0 := V^T Z V = \mathrm{diag}\,(z_1, \ldots, z_m)$, $z_1 \geq \cdots \geq z_m$ be Autonne decomposition of the symmetric $Z \in \mathbb{C}_{m\times m}$. So
$$X_0 := \begin{pmatrix} U^T & 0 \\ 0 & V^T \end{pmatrix} X \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \frac{1}{s_1}\begin{pmatrix} U^T S U & U^T A V \\ V^T A^T U & V^T Z V \end{pmatrix} = \frac{1}{s_1}\begin{pmatrix} S_0 & A'_0 \\ A'^T_0 & Z_0 \end{pmatrix}$$
is unitary, where $A'_0 := U^T A V$ and $S_0 := U^T S U = s_1 I_{n_1} \oplus s_2 I_{n_2} \oplus \cdots \oplus s_k I_{n_k}$. Now
$$X_0^* X_0 = I_{n+m}$$
which is equivalent to
$$S_0^2 + \overline{A'}_0 A'^T_0 = s_1^2 I_n \tag{6}$$
$$A'^*_0 A'_0 + Z_0^2 = s_1^2 I_m \tag{7}$$
$$S_0 A'_0 + \overline{A'}_0 Z_0 = 0_{n\times m}. \tag{8}$$
From (6),
$$A'_0 A'^*_0 = 0_{n_1\times n_1} \oplus (s_1^2 - s_2^2) I_{n_2} \oplus \cdots \oplus (s_1^2 - s_k^2) I_{n_k}, \tag{9}$$
and from (7),
$$A'^*_0 A'_0 = s_1^2 I_m - \mathrm{diag}\,(z_1^2, \ldots, z_m^2).$$
The eigenvalues of $A'_0 A'^*_0$ and $A'^*_0 A'_0$ are identical (counting multiplicities) except some zeros. So
$$Z_0 = s_1 I_h \oplus s_2 I_{n_2} \oplus \cdots \oplus s_k I_{n_k},$$
where $h := m - n + n_1$, and
$$A'^*_0 A'_0 = 0_{h\times h} \oplus (s_1^2 - s_2^2) I_{n_2} \oplus \cdots \oplus (s_1^2 - s_k^2) I_{n_k}. \tag{10}$$
It follows from (9) and (10) that
$$A'_0 = \begin{pmatrix} 0_{n_1\times h} & 0 \\ 0 & A_0 \end{pmatrix},$$
where $A_0 \in \mathbb{C}_{(n-n_1)\times(n-n_1)}$. By (8) one has
$$(s_2 I_{n_2} \oplus \cdots \oplus s_k I_{n_k}) A_0 + \overline{A}_0 (s_2 I_{n_2} \oplus \cdots \oplus s_k I_{n_k}) = 0_{(n-n_1)\times(n-n_1)}.$$
Notice that $s_2 > s_3 > \cdots > s_k$ so that by block multiplication, we have
$$A_0 = B_2 \oplus B_3 \oplus \cdots \oplus B_k,$$
where $B_j \in i\mathbb{R}_{n_j\times n_j}$, $j = 2, \ldots, k-1$. If $s_k \neq 0$, then $B_k \in i\mathbb{R}_{n_k\times n_k}$; if $s_k = 0$, then $B_k \in \mathbb{C}_{n_k\times n_k}$. Then by (9) or (10), A_0 is of the following diagonal block form
$$(s_1^2 - s_2^2)^{1/2} A_2 \oplus (s_1^2 - s_3^2)^{1/2} A_3 \oplus \cdots \oplus (s_1^2 - s_k^2)^{1/2} A_k,$$

where $iA_{n_j} \in \mathbb{R}_{n_j \times n_j}$ is an orthogonal matrix for $j = 2, \ldots, k$, except for iA_k when $s_k = 0$. If $s_k = 0$, then A_k is unitary.

Conversely, (5) is a unitary symmetric completion of S.

Clearly when $h = 0$, that is, $m = n - n_1$, X is smallest in size.

COROLLARY 2.2. Let $S \in \mathbb{C}_{n\times n}$ be a complex symmetric nonsingular matrix with singular values $s_1 > s_2 > \cdots > s_n > 0$ with Autonne decomposition $U^T SU = \operatorname{diag}(s_1, \ldots, s_n)$, where U is unitary. Then

$$X = \frac{1}{\|S\|}\begin{pmatrix} S & A \\ A^T & Z \end{pmatrix}$$

is a unitary symmetric matrix of smallest size if and only if $X \in \mathbb{C}_{(2n-1)\times(2n-1)}$ and

$$VZV^T = \operatorname{diag}(s_2, \ldots, s_n),$$

for some unitary $V \in \mathbb{C}_{(n-1)\times(n-1)}$, and

$$U^T AV = \begin{pmatrix} 0_{1\times(n-1)} \\ A_0 \end{pmatrix},$$

where

$$A_0 = \operatorname{diag}(\pm i(s_1^2 - s_2^2)^{1/2}, \cdots, \pm i(s_1^2 - s_n^2)^{1/2}) \in \mathbb{C}_{n\times(n-1)}.$$

REMARK 2.3. One can deduce the corrected version of Theorem 1.1 from Corollary 2.2. Of course, we assume $s_1 > s_2 > \cdots > s_n > 0$ due to Example 1.3. By Autonne decomposition,

$$S_0 := U^T SU = \operatorname{diag}(s_1, \ldots, s_n).$$

Then $S = \overline{U} S_0 U^*$ and $SS^* = \overline{U} S_0^2 \overline{U}^*$. Hence $\overline{u}_j$ is a unit eigenvector of SS^* corresponding to the eigenvalue s_j^2, $j = 1, \ldots, n$. Since all the eigenvalues of SS^* are simple, the unit eigenvector w_j, in Theorem 1.1 must be a scalar multiple of $\overline{u}_j$, say,

$$\overline{u}_j = e^{i\xi_j} w_j, \quad j = 2, \ldots, n.$$

One can easily recover ξ_j via $w_j^* S\overline{w}_j = e^{2i\xi} s_j$ so that

$$2\xi_j = \arg(w_j^* S\overline{w}_j), \quad j = 1, \ldots, n. \tag{11}$$

From this point of view, the vectors $\overline{u}_j$, $j = 1, \ldots, n$, are the canonical unit eigenvectors of SS^* with respect to transforming S into a diagonal matrix with a unitary congruence $U^T SU = S_0$. Let

$$\Lambda = \operatorname{diag}((s_1^2 - s_2^2)^{1/2}, \ldots, (s_1^2 - s_n^2)^{1/2}).$$

By Theorem 2.1, we may choose

$$A_0 = -i\,\Lambda, \quad V = \operatorname{diag}(e^{-i\phi_2/2}, \ldots, e^{-i\phi_n/2}).$$

so that $Z = \mathrm{diag}\,(s_2 e^{i\phi_2}, \ldots, s_n e^{i\phi_n})$. By Theorem 2.1,

$$\begin{aligned} A &= \overline{U} \begin{pmatrix} 0 \\ A_0 \end{pmatrix} V^* \\ &= [\overline{u}_2 \cdots \overline{u}_n] A_0 \,\mathrm{diag}\,(e^{i\phi_2/2}, \ldots, e^{i\phi_n/2}) \\ &= [w_2 \cdots w_n]\, \mathrm{diag}\,(e^{i(-\pi/2+\xi_2+\phi_2/2)}, \ldots, e^{i(-\pi/2+\xi_n+\phi_n/2)})\, \Lambda. \end{aligned}$$

Let $\theta_j := -\pi/2 + \xi_j + \phi_j/2$. Clearly $\phi_j = \pi - 2\xi_j + 2\theta_j$, $j = 1, \ldots, n$, that is, $\phi_j = \pi - \arg\,(w_j^* S \overline{w}_j) + 2\theta_j$ by (11). This is just (3).

3 Unitary Skew Symmetric Completions for Skew Symmetric Matrices

Of independent interest, we consider the unitary skew symmetric completion problem for a given complex skew symmetric matrix $S \in \mathbb{C}_{n\times n}$. The singular values of a complex skew symmetric matrix A occur in pairs, except 0 when n is odd [4, Problem 25, p.217]. Indeed according to Hua decomposition [2, Theorem 7, p.481], there exists a unitary matrix $U \in \mathbb{C}_{n\times n}$ such that

$$U^T S U = \begin{cases} s_1 J_{2n_1} \oplus s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k} & \text{if } n \text{ is even} \\ s_1 J_{2n_1} \oplus s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k} \oplus (0) & \text{if } n \text{ is odd,} \end{cases} \tag{12}$$

where $s_1 > s_2 > \cdots > s_k$ are the distinct eigenvalues of S and $J_{2p} \in \mathbb{C}_{2p\times 2p}$ is the following diagonal block matrix

$$J_{2p} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A unitary matrix $A \in \mathbb{C}_{2n\times 2n}$ is said to be *symplectic* if $A^T J_{2n} A = J_{2n}$. Notice that our definition is different from [3, p.445] due to a different choice of skew symmetric bilinear form, namely J_{2n}. Nevertheless, the two differ by a (fixed) permutation similarity. Our choice is made for easier presentation of the proof of the following theorem.

THEOREM 3.1. Let $S \in \mathbb{C}_{n\times n}$ be a nonzero complex skew symmetric matrix with Hua decomposition (12). Then

$$X = \frac{1}{\|S\|} \begin{pmatrix} S & A \\ -A^T & Z \end{pmatrix} \in \mathbb{C}_{(m+n)\times(m+n)}$$

is a unitary skew symmetric matrix if and only if

1. the singular values of Z are those of S, counting multiplicities, except s_1 whose multiplicity is $2h := m - n + 2n_1$, that is, there exists a unitary $V \in \mathbb{C}_{m\times m}$ such that

$$V^T Z V = \begin{cases} s_1 J_{2h} \oplus s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k} & \text{if } n \text{ is even} \\ s_1 J_{2h} \oplus s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k} \oplus (0) & \text{if } n \text{ is odd,} \end{cases}$$

(so n and m have the same parity), and

2.

$$U^TAV = \begin{cases} 0_{2n_1\times 2h} \oplus A_0 & \text{if } n \text{ is even} \\ 0_{2n_1\times 2h} \oplus A_0 \oplus (s_1) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$A_0 = (s_1^2 - s_2^2)^{1/2}A_2 \oplus (s_1^2 - s_3^2)^{1/2}A_3 \oplus \cdots \oplus (s_1^2 - s_k^2)^{1/2}A_k$$

such that $iA_j \in \mathbb{C}_{2n_j\times 2n_j}$, $j = 2, \ldots, k$, are symplectic matrices except for iA_k when $s_k = 0$. When $s_k = 0$, A_k is unitary.

When $h = 0$, that is, $m = n - 2n_1$, X is smallest in size.

PROOF. We now provide a proof when n, the size of S, is even. The odd case is similar. Let

$$S_0 := U^TSU = s_1J_{2n_1} \oplus s_2J_{2n_2} \oplus \cdots \oplus s_kJ_{2n_k},$$

and

$$Z_0 := V^TZV = \begin{cases} z_1J_{2m_1} \oplus z_2J_{2m_2} \oplus \cdots \oplus z_\ell J_{2m_\ell} & \text{if } m \text{ is even} \\ z_1J_{2m_1} \oplus z_2J_{2m_2} \oplus \cdots \oplus z_\ell J_{2m_\ell} \oplus (0) & \text{if } m \text{ is odd} \end{cases}$$

be Hua decompositions of the skew symmetric $S \in \mathbb{C}_{n\times n}$ and $Z \in \mathbb{C}_{m\times m}$, where $z_1 \geq \cdots \geq z_\ell$. Clearly X is unitary if and only if

$$X_0 := \begin{pmatrix} U^T & 0 \\ 0 & V^T \end{pmatrix} X \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \frac{1}{s_1}\begin{pmatrix} S_0 & {A'}_0 \\ -{A'}_0^T & Z_0 \end{pmatrix}$$

is unitary, where $A'_0 = U^TAV \in \mathbb{C}_{n\times m}$. It amounts to $X_0^*X_0 = I_{n+m}$, that is,

$$-S_0^2 + \overline{A'}_0{A'}_0^T = s_1^2 I_n \tag{13}$$

$${A'}_0^*A'_0 - Z_0^2 = s_1^2 I_m \tag{14}$$

$$-S_0A'_0 - \overline{A'}_0Z_0 = 0_{n\times m}. \tag{15}$$

Now (13) and (14) yield

$$\begin{aligned} A'_0{A'_0}^* &= 0_{2n_1} \oplus (s_1^2 - s_2^2)I_{2n_2} \oplus \cdots \oplus (s_1^2 - s_k^2)I_{2n_k}, \\ {A'_0}^*A'_0 &= \begin{cases} s_1^2I_m - [z_1^2I_{2m_1} \oplus z_2^2I_{2m_2} \oplus \cdots \oplus z_\ell^2 I_{2m_\ell}] & \text{if } m \text{ is even} \\ s_1^2I_m - [z_1^2I_{2m_1} \oplus z_2^2I_{2m_2} \oplus \cdots \oplus z_\ell^2 I_{2m_\ell} \oplus (0)] & \text{if } m \text{ is odd,} \end{cases} \end{aligned}$$

where $z_1 \geq \cdots \geq z_\ell$. Because the singular values of ${A'_0}^*A_0$ and $A'_0{A'_0}^*$ are identical (counting multiplicities) except for some zeros, m must be even and

$$Z_0 = s_1J_{2h} \oplus s_2J_{2n_2} \oplus \cdots \oplus s_kJ_{2n_k},$$

$2h := m - n - 2n_1$. Hence

$$A'_0 = \begin{pmatrix} 0_{2n_1\times 2h} & 0 \\ 0 & A_0 \end{pmatrix},$$

where $A_0 \in \mathbb{C}_{(n-2n_1)\times(n-2n_1)}$ and

$$A_0 A_0^* = A_0^* A_0 = (s_1^2 - s_2^2) I_{2n_2} \oplus \cdots \oplus (s_1^2 - s_k^2) I_{2n_k}. \tag{16}$$

Now (15) implies that

$$(s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k})^T A_0 - \overline{A}_0 (s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k}) = 0. \tag{17}$$

Multiplying both sides of (17) by $s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k}$ from the left, we have

$$(s_2^2 I_{2n_2} \oplus \cdots \oplus s_k^2 I_{2n_k}) A_0 - (s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k}) \overline{A}_0 (s_2 J_{2n_2} \oplus \cdots \oplus s_k J_{2n_k}) = 0.$$

Since $s_2 > \cdots > s_k$, by straightforward computation,

$$A_0 = (s_1^2 - s_2^2)^{1/2} A_2 \oplus (s_1^2 - s_3^2)^{1/2} A_3 \oplus \cdots \oplus (s_1^2 - s_k^2)^{1/2} A_k, \tag{18}$$

where $A_j \in \mathbb{C}_{2n_j \times 2n_j}$, $j = 2, \ldots, k$. By (16), each A_j is a unitary matrix. Substituting (18) into (17) yields

$$J_{2n_j} A_j + \overline{A}_j J_{2n_j} = 0, \quad j = 2, \ldots, k,$$

except for A_k when $s_k = 0$. Since A_j is unitary, iA_j is symplectic, $j = 1, \ldots, k$ except A_k when $s_k = 0$. More precisely, A_j is of the following 2×2 block form $A_j = (A_{st})_{n_j \times n_j}$:

$$A_{st} = \begin{pmatrix} a_{st} & b_{st} \\ \overline{b}_{st} & -\overline{a}_{st} \end{pmatrix}, \quad a_{st}, b_{st} \in \mathbb{C},$$

except for A_k when $s_k = 0$.

Conversely it is easy to see that (5) is a unitary completion of S.

Clearly X is smallest in size if $h = 0$.

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